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RISE OVER RUN OR RATE OF CHANGE? EXPLORING AND EXPANDING STUDENT UNDERSTANDING OF SLOPE IN ALGEBRA I.

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Student understanding of slope and rate of change is often formulaic and underdeveloped. This presents problems for students in secondary and post-secondary mathematics where slope and rate of change are key foundational concepts. To study how students develop robust understandings of slope and rate of change, we conducted a design experiment in a U.S. high school Algebra I classroom that focused on developing versatile and adaptable knowledge of slope using rate of change as a foundational concept for slope. In this workshop, participants will contribute to an international perspective on the teaching of slope, engage in key activities that were used in the design experiment, and explore student work generated from these activities.

TOWARD THE DEVELOPMENT OF ROBUST UNDERSTANDINGS OF SLOPE AND RATE OF CHANGE

In the United States, the concept of "slope" is typically introduced to students in middle school before they officially enroll in Algebra I (generally in their first year of high-school). However, student understanding of this concept is often formulaic and underdeveloped. In using the term "underdeveloped", we mean to say that student understanding is not "robust" in the sense that they can use slope in versatile and adaptable ways (we define and describe these terms in more detail below). Furthermore, when prompted for explanation, a student is likely to describe slope as "rise over run" or "change in y over change in x." Neither is incorrect, but both leave to question if the student is simply repeating a definition or if the student truly has robust, conceptual understanding that can be generalized and transferred across multiple contexts. Such conceptual understanding is important because the concept of slope will come to underlie much of the curriculum in Algebra I. Moreover, the concept of slope is arguably the most important concept in Calculus (Stump, 2001; Thompson, 1994a).

To investigate how students develop robust understandings of slope, we conducted a 4-week design experiment in a U.S. high school Algebra I classroom. Following Sfard & Linchevski (1994), we define "robust understanding" to be composed of *versatility* (the ability to hold multiple perspectives) and *adaptability* (the ability to bring the appropriate perspectives to bear on a particular problem). Applying this definition to slope, we considered the following seven sub-constructs (Stump, 1999):

- Slope as a geometric ratio, e.g., "rise over run"
- Slope as an algebraic ratio or formula, e.g., "change in y over change in x"
- Slope as a physical property, e.g., "steepness"
- Slope as a functional property, e.g., "rate of change"
- Slope as a parametric coefficient, e.g., the a in the equation, $y = ax + b$
- Slope as a trigonometric ratio, that is, the tangent of the angle that a linear graph makes with the x-axis
- Slope as the derivative of a function

In the U.S., the first five sub-constructs are generally learned in Algebra I (the remaining sub-constructs are learned in Algebra II, pre-calculus, and calculus). A "robust understanding" of slope therefore involves the versatility of knowing the first five sub-constructs, and the adaptability of choosing which sub construct is appropriate to solve a particular problem. Many researchers (Bransford, Brown, & Cocking, 2000; Kaput, 1999) suggest that if knowledge is to be versatile and adaptable, it should be "organized" around a key concept, and that this key concept should be developed from students' informal understanding.

Which of the above sub-constructs should be the key concept? The preponderance of professional organizations and research literature argues for the primacy of the functional property. For example, the National Council of Mathematics Teachers considers rate of change to be one of five foundational ideas in mathematics:

> Foundational ideas like place value, equivalence, proportionality, function, and *rate of change* should have a prominent place in the mathematics curriculum because they enable students to understand other mathematical ideas and connect ideas across different areas of mathematics. (NCTM, 2000, p. 11, emphasis added)

The call for the primacy of the functional property is echoed by researchers who have studied student learning in calculus (Stroup, 2002; Thompson, 1994a). For example, Thompson (1994a) argued that "concepts of rate of change… are central to understanding the Fundamental Theorem [of Calculus] (p. 229)."

Finally, the functional property can be built upon students' informal reasoning. Stump (2001) found that students were more successful using informal reasoning when solving problems involving rate of change than they were when solving problems involving steepness, and Confrey & Smith (1994) found that the concept of rate was accessible to even very young children.

A DESIGN EXPERIMENT TO STUDY THE DEVELOPMENT OF ROBUST UNDERSTANDINGS OF SLOPE AND RATE OF CHANGE

The curriculum we designed during the experiment followed a hypothetical learning trajectory that helped us make "a prediction as to the path by which learning might proceed" (Simon, 1995, p. 135). The hypothetical learning trajectory we started with is as follows:

- Students learn rate of change first through proportional reasoning, and then as a measure of covariation (based on the work of Confrey & Smith, 1994; Cramer, Bezuk, & Behr, 1989; Karplus, Pulos, & Stage, 1983; Nemirovsky, 1996; Nunes, Desli, & Bell, 2003; Thompson, 1994b; Tierney & Monk, 2007a; and Yerushalmy, 1997). Through this process, students construct the parametric coefficient as a way of using rates of change to make predictions.
- Using multiple representations of functions (tables, graphs, and verbal descriptions of realistic situations), students construct the algebraic ratio and geometric ratio as ways to calculate a rate of change (this extends the work of Brenner et al. [1997]).
- By examining the relationship between the rate of change and the shape of a linear graph, students construct the physical property as a measure of steepness (based on the work of Lobato & Thanheiser, 2002 and Tierney & Monk, 2007).

Figure 1 shows a schematic of this learning trajectory, including sample activities.

By the end of the design experiment, we hoped to see that students had the versatility of knowing the multiple sub-constructs, and the adaptability of knowing which sub-construct to use in particular situations. Specifically, we hoped that students would understand that the functional property and physical property are both used as measures (of covariation and steepness, respectively), that the algebraic and geometric ratios are ways to calculate these measurements, and that the parametric coefficient can be interpreted as a functional property to make predictions.

Because meaning-making is an emergent, constructive process (Cobb, 2000), the "actual" trajectory was constantly modified as we assessed the emergent understandings in the classroom. Such analysis informs theory, which in turn informs practice. As stated by Gravemeijer (1994, p. 449):

> What is invented behind the desk is immediately put into practice; what happens in the classroom is consequently analysed, and the result of this analysis is used to continue the developmental work.

WORKSHOP DESCRIPTION

In this workshop, participants will engage in mathematical activity and conversation. In particular, participants will share how slope is taught in their country and how language is used to talk about slope. Through this discussion, participants will contribute to an international perspective on the teaching and development of slope. The presenters will share how their work on student understandings of slope is relevant to the Common Core State Standards (CCSS) for Mathematics that have recently gained prominence in the United States.

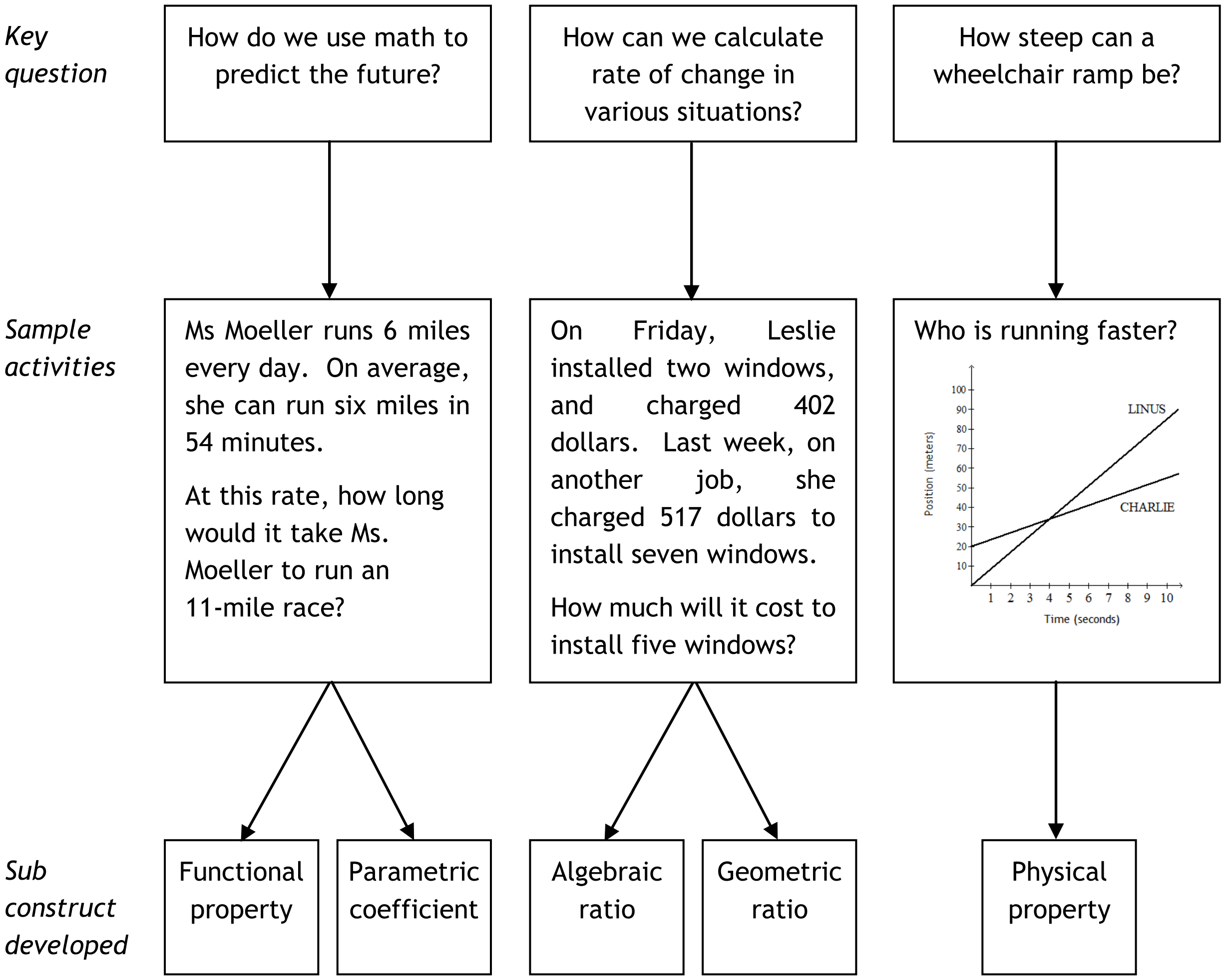

Figure 1. A schematic of our hypothetical learning trajectory

Then, participants will engage in some of the mathematical activities used during our design experiment and have a chance to discuss the mathematics entailed in these activities. Participants will also watch video clips from our experiment and have the opportunity to discuss student learning of slope and their ideas. The workshop will conclude with a discussion of next steps and implications for future research.

References

Bransford, J. D., Brown, A. L., & Cocking, R. R. (2000). *How People Learn: Brain, Mind, Experience, and School: Expanded Edition* (Vol. 24). National Academies Press.

Brenner, M. E., Mayer, R. E., Moseley, B., Brar, T., Durán, R., Reed, B. S., Webb, D., et al. (1997). Learning by Understanding: The Role of Multiple Representations in Learning Algebra. *American Educational Research Journal*, *34*(4), 663-689.

Cobb, P. (2000). Conducting Teaching Experiments in Collaboration with Teachers. In A. E. Kelly & R. Lesh (Eds.), *Research Design in Mathematics and Science Education* (pp. 307-333). Mahwah, NJ: Lawrence Erlbaum.

Confrey, J., & Smith, E. (1994). Exponential Functions , Rates of Change, and the Multiplicative Unit. *Educational Studies in mathematics*, *26*(2), 135–164.

Cramer, K., Bezuk, N., & Behr, M. J. (1989). Proportional relationships and unit rates. *Mathematics Teacher*, *82*(7), 537–544.

Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education*, *25*(5), 443–471.

Kaput, J. J. (1999). Teaching and Learning a New Algebra. In E. Fennema & T. A. Romberg (Eds.), *Mathematics Classrooms that Promote Understanding* (pp. 133-155). Mahwah, NJ: Erlbaum.

Karplus, R., Pulos, S., & Stage, E. K. (1983). Early Adolescents' Proportional Reasoning on "Rate" Problems. *Educational Studies in Mathematics*, *14*(3), 219–233.

Lobato, J., & Thanheiser, E. (2002). Developing understanding of ratio-as-measure as a foundation for slope. In B. H. Litwiller (Ed.), *Making Sense of Fractions, Ratios, and Proportions* (pp. 162-175). Reston, VA: NCTM.

NCTM. (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Nemirovsky, R. (1996). Mathematical narratives, modeling, and algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to Algebra: Perspectives for Research and Teaching* (pp. 197–220). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Nunes, T., Desli, D., & Bell, D. (2003). The development of children's understanding of intensive quantities. *International Journal of Educational Research*, *39*(7), 651-675.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification - The case of algebra. *Educational Studies in Mathematics*, *26*, 191-228.

Simon, M. A. (1995). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. *Journal for Research in Mathematics Education*, *26*(2), 114.

Stroup, W. M. (2002). Understanding qualitative calculus: a structural synthesis of learning research. *International Journal of Computers for Mathematical Learning*, 167-215.

Stump, S. L. (1999). Secondary mathematics teachers' knowledge of slope. *Mathematics Education Research Journal*, *11*(2), 124-144.

Stump, S. L. (2001). High school precalculus students' understanding of slope as measure. *School Science and Mathematics*, *101*(2), 81-89.

Thompson, P. W. (1994a). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, *26*(2), 229- 274.

Thompson, P. W. (1994b). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 179–234). Albany, NY: SUNY Press.

Tierney, C., & Monk, S. (2007). Children's reasoning about change over time. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 185-200). Lawrence Erlbaum.

Yerushalmy, M. (1997). Mathematizing Verbal Descriptions of Situations: A Language to Support Modeling. *Cognition and Instruction*, *15*(2), 207-264.