

# Making Algebra Meaningful

Fred Peck

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[www.RMEintheclassroom.com](http://www.RMEintheclassroom.com)

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Billings, MT  
October 18, 2018



# Agenda

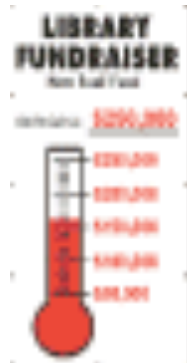
Do some math

Talk about what we did

# On the one hand...

## Lived experiences

- Contain mathematical principles
- Context-bound
- Models of a situation



# On the other hand...

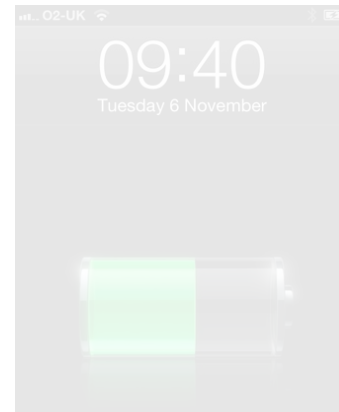
## Formal mathematics

- Potentially very general
- Far removed from context

$$\frac{3}{4}$$

## Lived experiences

- Contain mathematical principles
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- Models of a situation

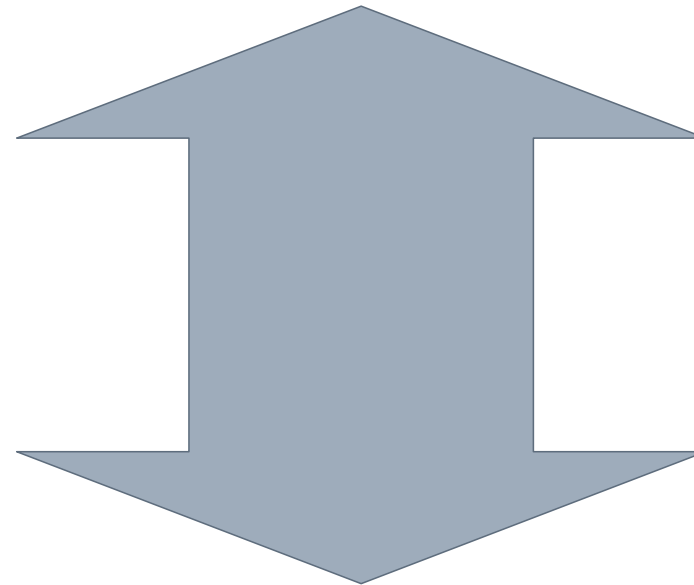




## Formal mathematics

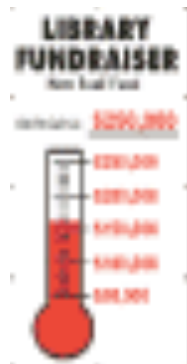
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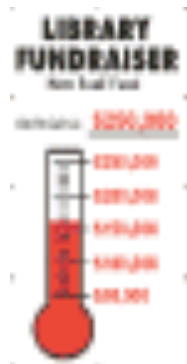
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How to “connect” formal mathematics with students’ lived experiences?

## Lived experiences

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# “traditional” sequence

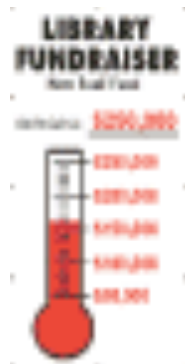
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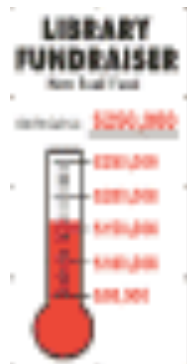
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# “traditional” sequence

- Structure without structuring
- Mathematics is disconnected from lived reality
- Math is seen as meaningless
- Little opportunity to participate in mathematical practices

# “discovery” sequence

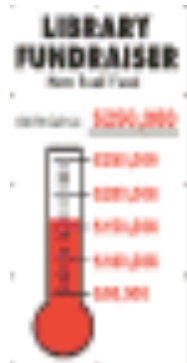
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# “discovery” sequence

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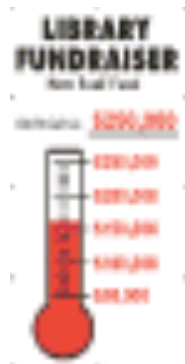
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# “discovery” sequence

Better! But still...

- Not enough structuring
- There is a big jump from lived experiences to formal mathematics – often too big
- Ultimately, formal mathematics is the only tool that students have to solve problems



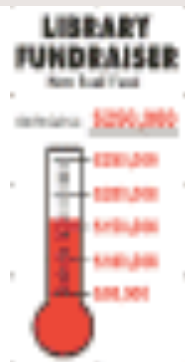
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# the “pre-formal” layer

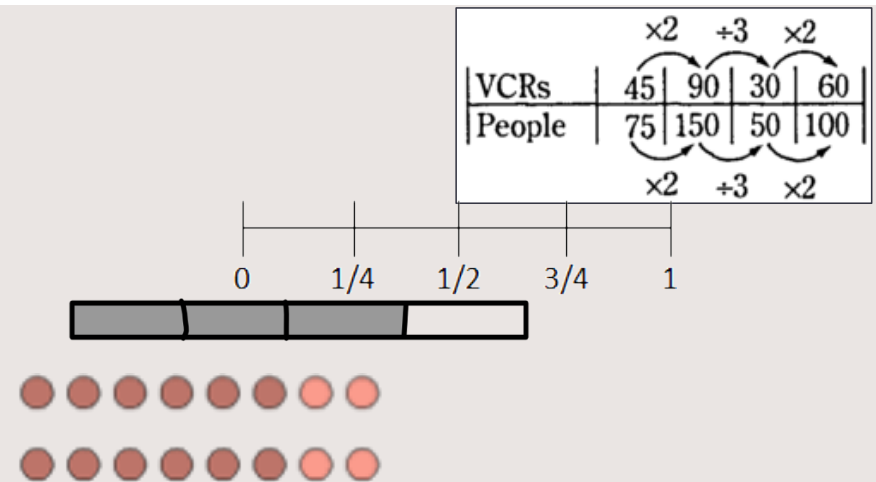
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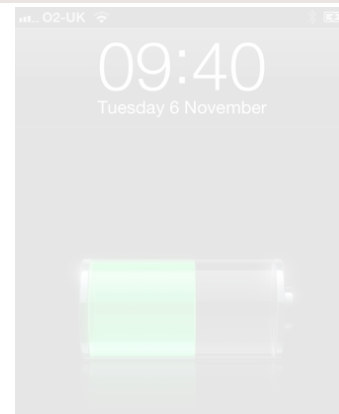
## “Pre-formal” models

- Generalizable, but still retain contextual cues
- Models for mathematics



## Lived experiences

- Contain mathematical principles
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# the iceberg model

## Formal mathematics

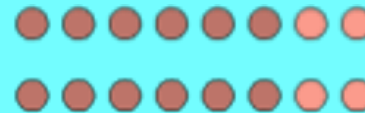
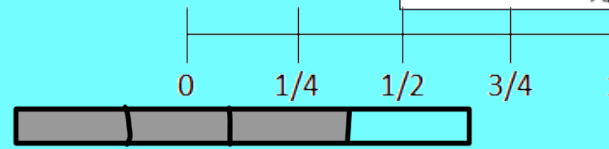
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## “Pre-formal” models

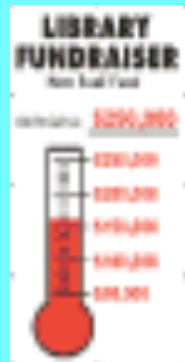
- Generalizable, but still retain contextual cues
- Models for mathematics

	$\times 2$	$+3$	$\times 2$	
VCRs	45	90	30	60
People	75	150	50	100
	$\times 2$	$+3$	$\times 2$	



## Lived experiences

- Contain mathematical principles
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# the “pre-formal” layer

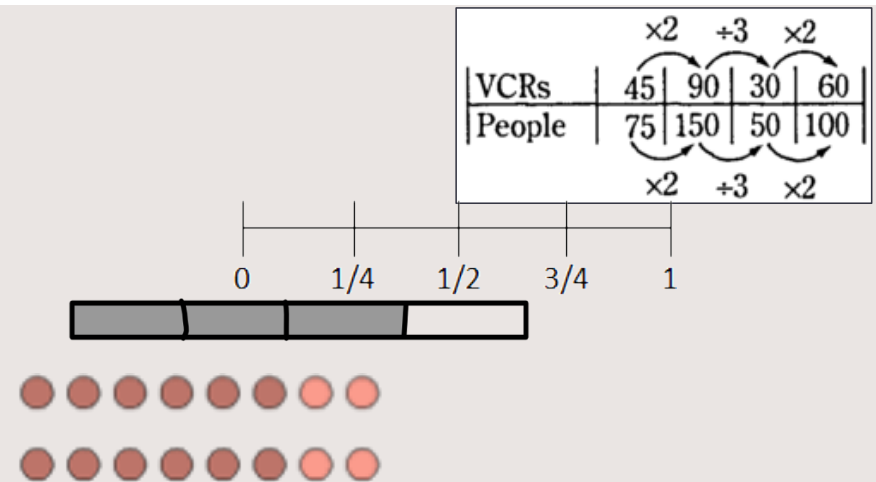
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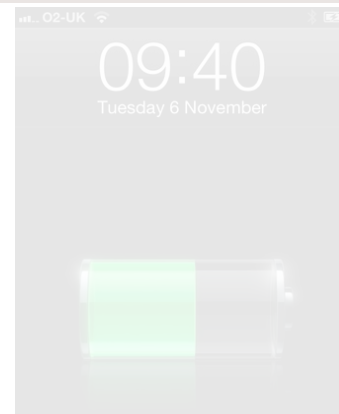
## “Pre-formal” models

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## Lived experiences

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# “Pre-formal” models

... help students *learn* mathematics

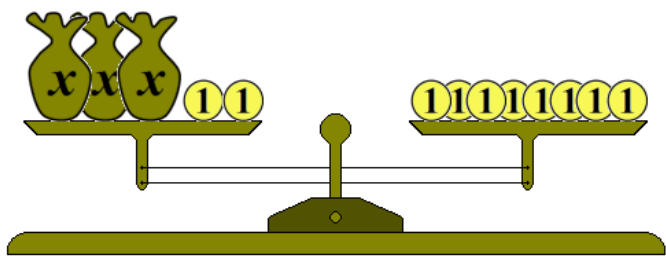
**Structure and meaning**  
of algebra equations

... are tools that students can use to *do* math

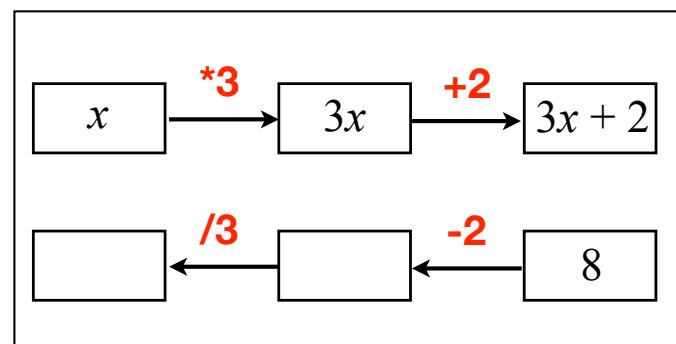
**Strategies** to solve  
algebra equations

# Two models for algebra equations

$$3x + 2 = 8$$



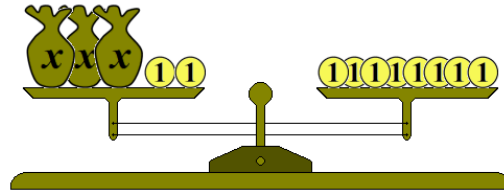
Balance model



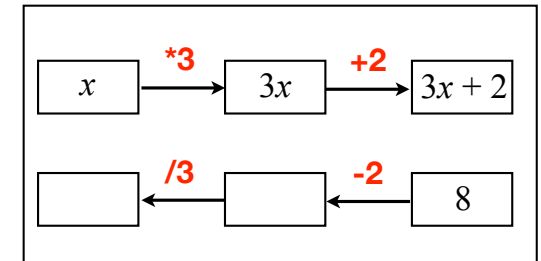
Arrow chain model

$$3x + 2 = 8$$

## Models



Balance model



Arrow chain model

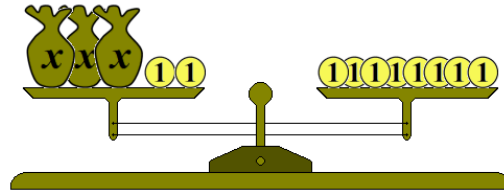
## Structure and meaning

The equation represents  
**objects**  
that are  
**grouped & compared**

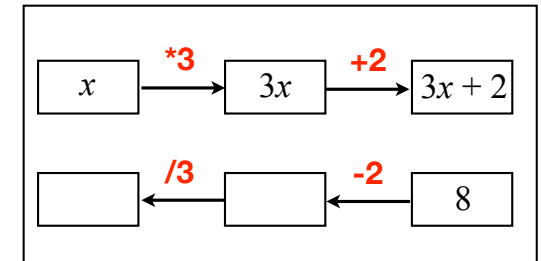
The equation represents  
**a process**  
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$$3x + 2 = 8$$

## Models



**Balance model**



**Arrow chain model**

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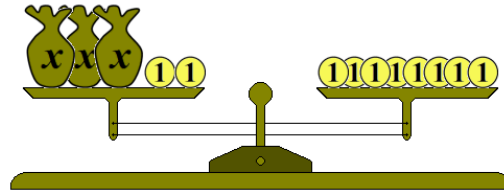
*"Three x's and two ones are  
the same as eight ones."*

*"Start with a number. Multiply  
it by three. Add two. You end  
up with 8. What number did  
you start with?"*

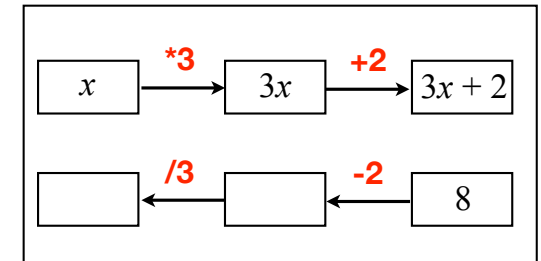


$$3x + 2 = 8$$

## Models



**Balance model**



**Arrow chain model**

## Structure and meaning

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## Strategy

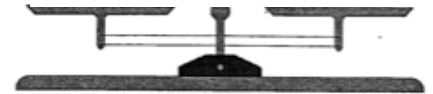
**Balance:**  
Make simpler equations  
by doing the same thing  
to both sides

**Backtrack:**  
Work backwards from  
the end, using opposite  
operations to undo.

# What do “pre-formal” models do for students?

1. Use the balance scale on the right to model the equation,

$$2x + 9 = 5x + 2$$



Models

# What do “pre-formal” models do for students?

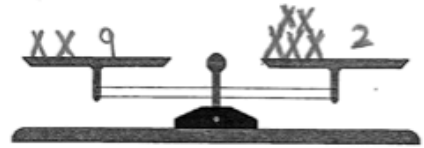
Models



Structure  
and  
meaning

1. Use the balance scale on the right to model the equation,  
 $2x + 9 = 5x + 2$

$x = x$   
( $x$  is displayed as an object)



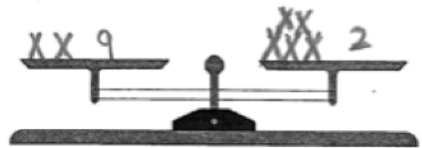
# What do “pre-formal” models do for students?

Models

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Strategy

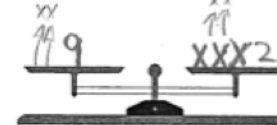
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$x = x$   
( $x$  is displayed as an object)

2. Then, use balance strategies to solve for  $x$ . For each step, draw the balance scale, write the equation, and explain your reasoning.

Step 1:



Equation:  $9 = 3x + 2$

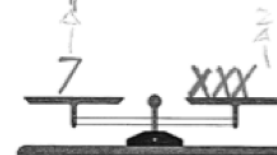
What did you do?

$-2x$  on both sides

Why will the scale remain balanced?

I did the same to both sides.

Step 2:



Equation:  $7 = 3x$

What did you do?

$-2$  from both sides

Why will the scale remain balanced?

I did the same to both sides.

Step 3:



Equation:  $3/7 = x$

What did you do?

$\div 3$  to both sides

Why will the scale remain balanced?

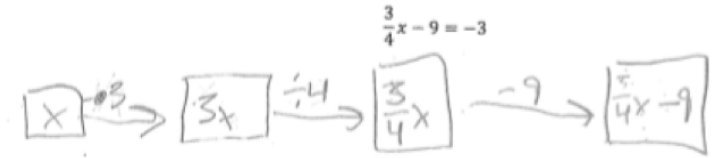
I did the same problem to both sides evenly.

# What do “pre-formal” models do for students?

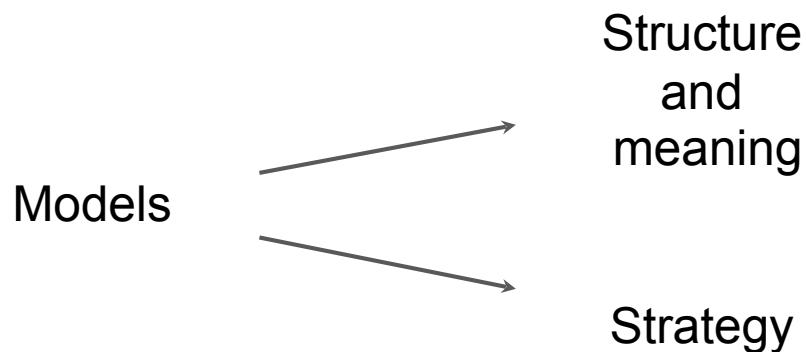
Models



Structure  
and  
meaning


$$\begin{aligned} & \boxed{x} \xrightarrow{+3} \boxed{3x} \xrightarrow{\div 4} \boxed{\frac{3}{4}x} \xrightarrow{-9} \boxed{\frac{3}{4}x - 9} \\ & \frac{3}{4}x - 9 = -3 \end{aligned}$$

# What do “pre-formal” models do for students?



The image shows two handwritten mathematical models for solving the equation  $\frac{3}{4}x - 9 = -3$ .

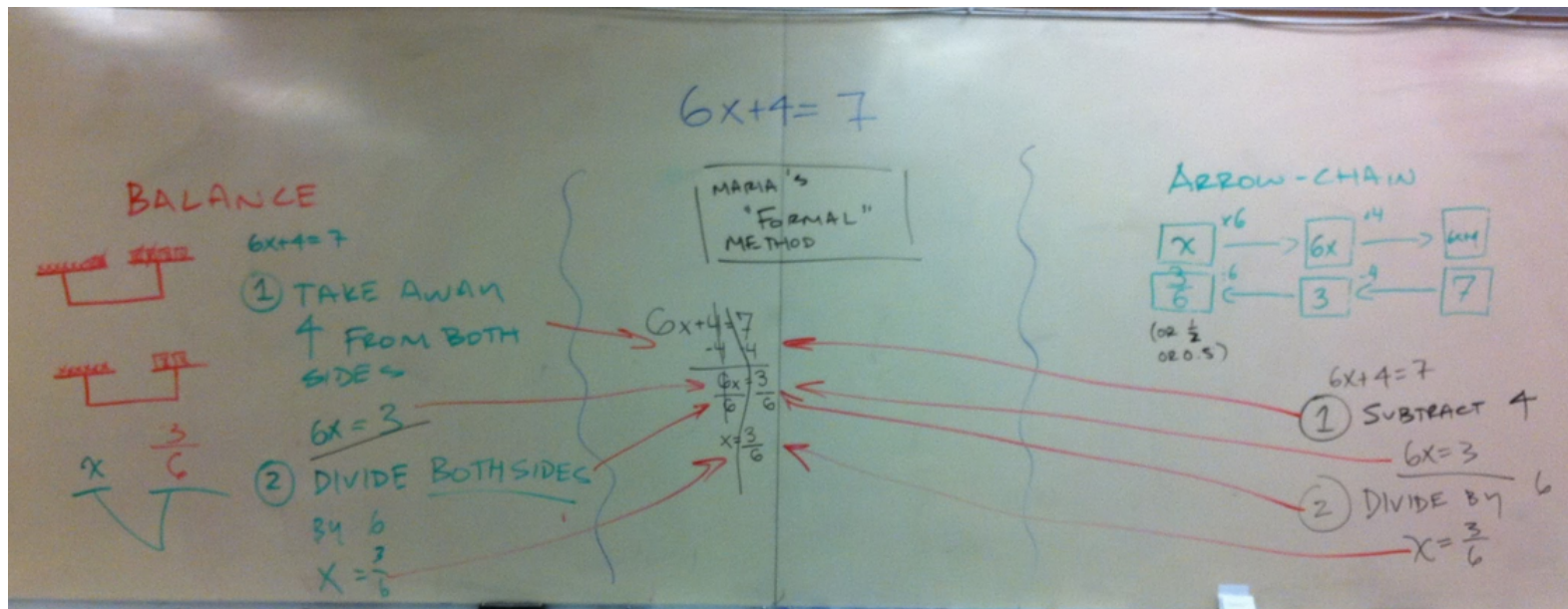
The top model is a forward process starting with  $x$  in a box. An arrow labeled  $\times 4$  points to a box containing  $3x$ . Another arrow labeled  $\div 4$  points to a box containing  $\frac{3}{4}x$ . A final arrow labeled  $-9$  points to a box containing  $\frac{3}{4}x - 9$ . Above the third box, the equation  $\frac{3}{4}x - 9 = -3$  is written.

The bottom model is a backward process starting with  $-3$  in a box. An arrow labeled  $+9$  points to a box containing  $6$ . Another arrow labeled  $\div 4$  points to a box containing  $24$ . A final arrow labeled  $\div 3$  points to a box containing  $8$ .

Models help students *learn* math

Models are tools that students can use to *do* math

# What do “pre-formal” models do for students?



Models help students *learn* math

Models are tools that students can use to *do* math



$$6x + 4 = 7$$

## BALANCE

$$6x + 4 = 7$$

- ① TAKE AWAY  
4 FROM BOTH  
SIDES

$$6x = 3$$

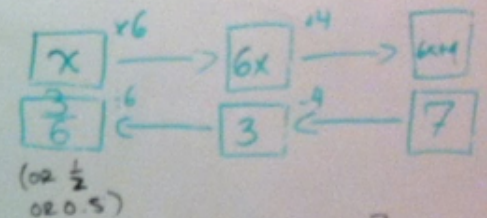
- ② DIVIDE BOTH SIDES

$$\begin{array}{r} 3 \cancel{4} \quad 6 \\ x = \frac{3}{6} \end{array}$$

## MARIA'S "FORMAL" METHOD

$$\begin{array}{r} 6x + 4 = 7 \\ -4 \quad -4 \\ \hline 6x = 3 \\ \div 6 \quad \div 6 \\ \hline x = \frac{3}{6} \end{array}$$

## ARROW-CHAIN



$$6x + 4 = 7$$

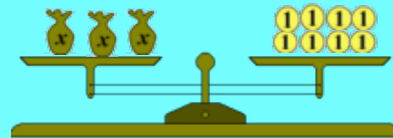
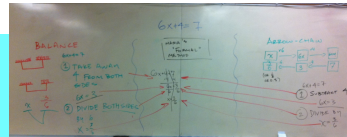
- ① SUBTRACT 4  
 $6x = 3$   
② DIVIDE BY 6  
 $x = \frac{3}{6}$

Models help students *learn* math

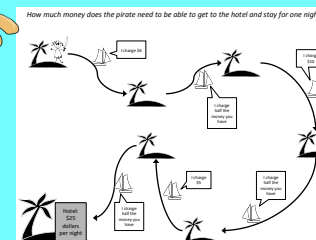
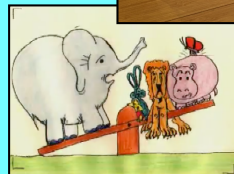
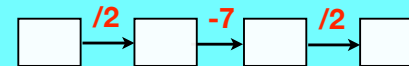
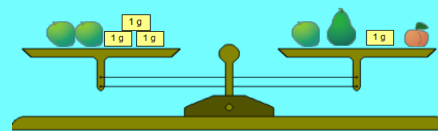
Models are tools that students can use to *do* math



$$3x + 2 = 8$$



Words	Arrow chain	Algebra equation	Solution
Start with a number	$x - 2 \rightarrow -6$	$2x - 6 = 4$	$x = 5$
Multiply by 2	$\rightarrow \rightarrow 4$		
Subtract 6			
End up with 4			



# Instructional sequences

## Connect to formal

- Squiggle between models and formal

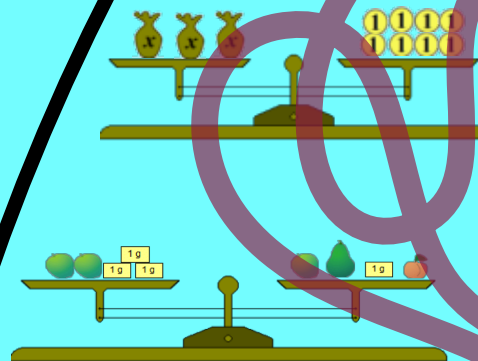
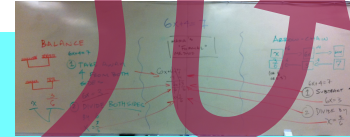
## The “pre-formal” layer

- Spend lots of time solving problems using models
- Goal: Models become “tools to think with”

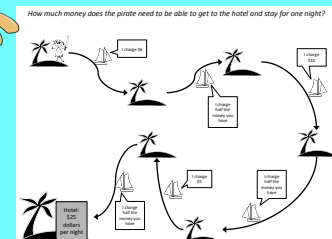
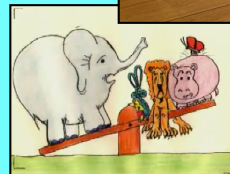
## Start

- Activities are “experientially real” for students
- Informal models of the situation can become “preformal” models *for* mathematics

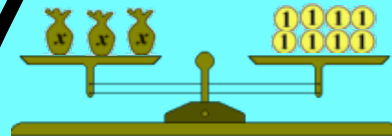
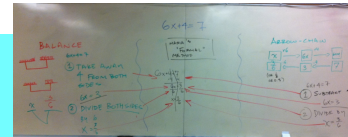
$$3x + 2 = 8$$



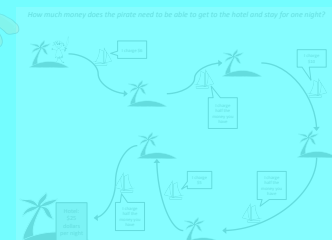
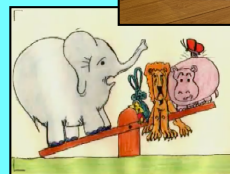
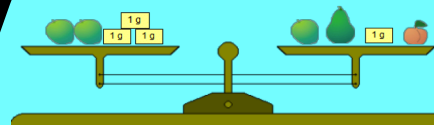
Words	Arrows	Algebra equation	Solution
Start with a number	$x$		
Multiply by 2	$\times 2$	$2x$	
Subtract 6	$- 6$	$2x - 6 = 4$	
End up with 4			$x = 5$

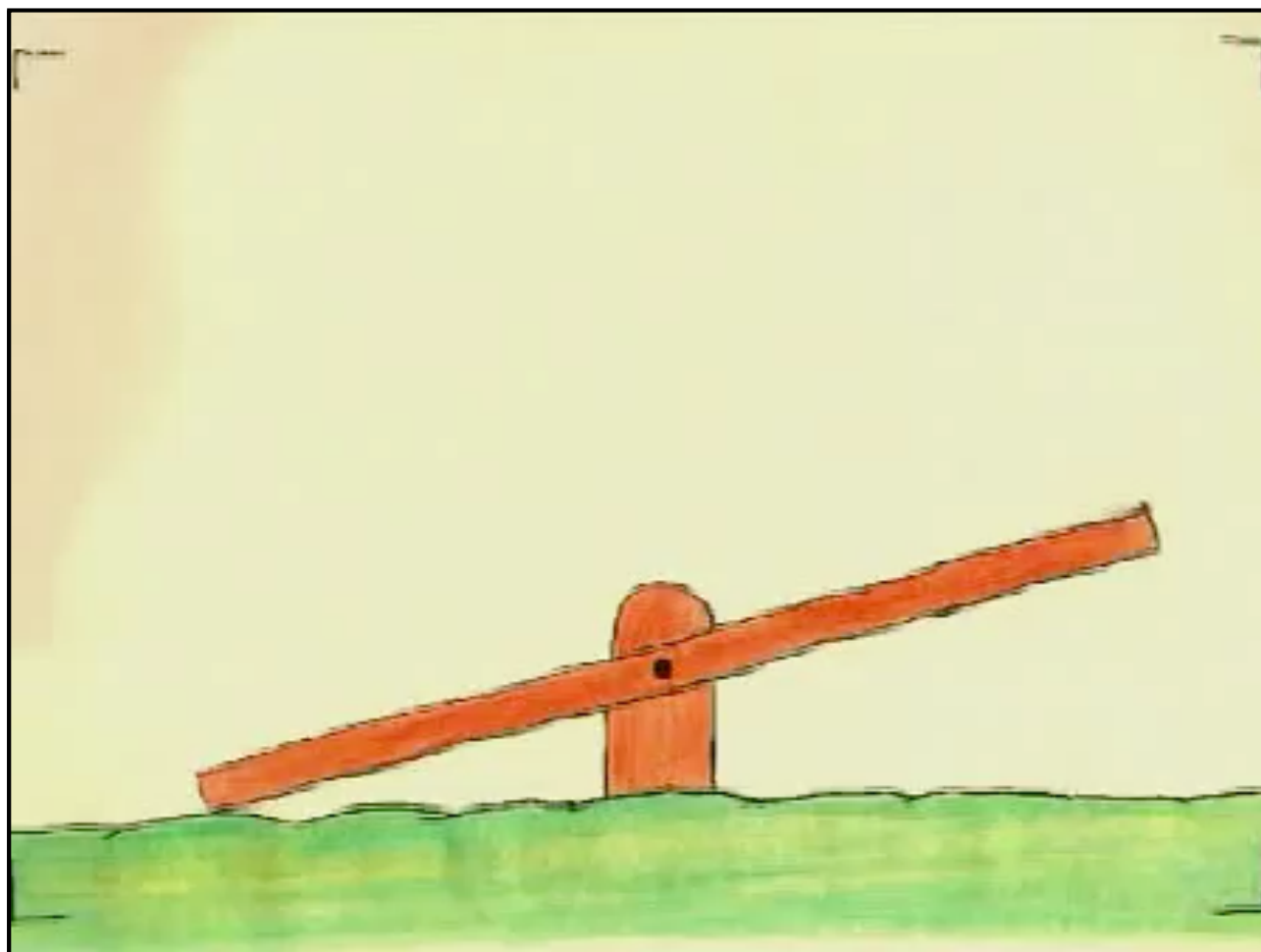


$$3x + 2 = 8$$



Start	Operation	Result	Operation	Result
Start with 6				
Divide by 2	$\div 2$	3		
Subtract 4	$- 4$	-1		
End up with 4				







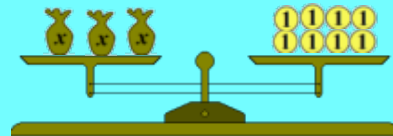
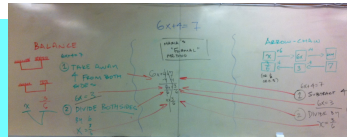


One piece of fruit can be weighed directly

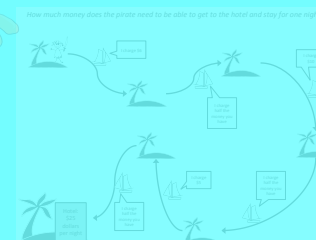
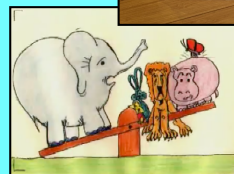
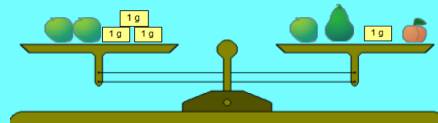
One piece can't be weighted directly, but multiple pieces can.

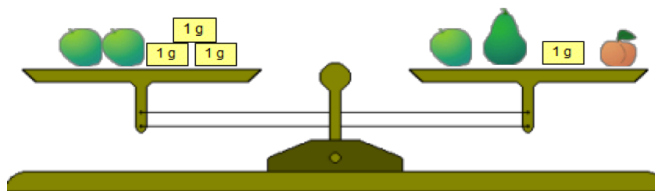
Need to use weights on both sides

$$3x + 2 = 8$$



Start	Process	Reverse Process	Result
Start with a number	1 → 2 → 3	3 → 2 → 1	
Multiply by 2	1 → 2	2 → 1	
Subtract 5	1 → 6	6 → 1	
Add up with 4	1 → 5	5 → 1	





#### PROBLEM ONE: RICE

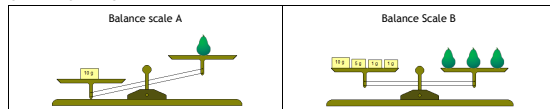
Imagine that you sell bananas. You use a balance scale to weigh the bananas. You have (only) the following weights available:

Two 1g weights 	One 5g weight 	One 10g weight 	One 50g weight 
--------------------	-------------------	--------------------	--------------------

A customer comes in and wants to buy 39g of bananas. On the balance scale below, show how you can weigh 39g of bananas.



#### PROBLEM TWO: PEARS



Which balance scale can you use to find the weight of one pear: Balance Scale A or Balance Scale B? Explain why you chose the scale that you did

#### PROBLEM THREE: KEEPING THE BALANCE

This scale is balanced:



What would the scale look like if you added 1g on the left side? Draw a picture below:

Describe two ways that you could make the scale balance again.

#### PROBLEM FOUR: IS IT BALANCED?

For each pair of scales below, the top scale is balanced. Will the bottom scale be balanced? How do you know?

(a) This scale is balanced

Will this be balanced?

How do you know?

(b) This scale is balanced

Will this be balanced?

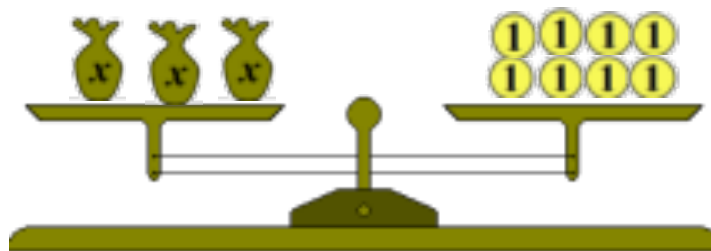
How do you know?

#### CHALLENGE PROBLEM

Go back to problem 1. Make a list of all the banana weights that you can measure using your weights. How do you know that you have all of them?







**PROBLEM FIVE: EQUATIONS ON THE BALANCE SCALE**

1. Use the balance scale on the right to model the equation,  
 $3x + 2 = 8$



2. Then, use balance strategies to solve for  $x$ . For each step, draw the balance scale, write the equation, and explain your reasoning.

Step 1:

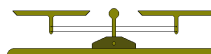


Equation: \_\_\_\_\_

What did you do?

Why will the scale remain balanced?

Step 2:



Equation: \_\_\_\_\_

What did you do?

Why will the scale remain balanced?

Step 3:



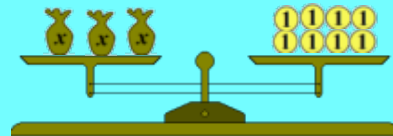
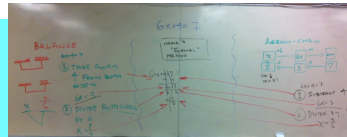
Equation: \_\_\_\_\_

What did you do?

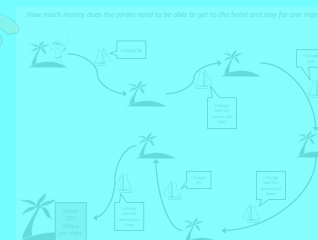
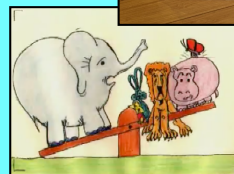
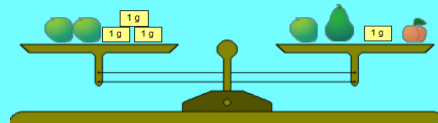
Why will the scale remain balanced?

$x =$  \_\_\_\_\_

$$3x + 2 = 8$$



Start	Process	Reverse Process	Result
Start with a number	1 → 2 → 3	3 → 2 → 1	
Multiply by 2	1 → 2	2 → 1	
Subtract 5	1 → 4	4 → 1	
Add up with 4	1 → 5	5 → 1	



# Extensions

Draw an arrow chain for each equation

$$16x^2 - 8x + 11 = 9$$

$$(4x - 2)^2 + 7 = 9$$

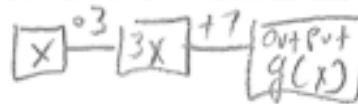
# Extensions

3. Consider the function,  $g(x) = 3x + 7$

a. Write a real-world situation for this function.

*You started with \$7 & get \$3 every day.*

b. Make an arrow chain for this function.



c. Complete the table below

Input $x$	Output $g(x)$
-3	-2
-1	4
2	13
5	22

Arrow chains for functions

# Extensions

Consider the balance scales on the left. Find the weight of one lemon and one pineapple using a *substitution* strategy.

For each step, draw the balance scales and explain your method.

**Step 1:**

What did you do?  
I subtracted 19 from both sides of top scale.

Why will the scale remain balanced?  
I made even trades.

**Step 2:**

What did you do?  
I substituted a pineapple for 2 lemons.

Why will the scale remain balanced?  
they were equal.

Balance models for systems of equations

# Thank-you!

Fred Peck

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## MCTM trivia night:

Craft Local  
7pm doors  
8:15 Trivia!