Beyond rise over run: A local instructional theory for slope

Frederick Peck

Freudenthal Institute US, University of Colorado

Frederick.Peck@Colorado.edu

Working version. Please do not distribute or cite without permission.

Abstract

In this paper I present a local instructional theory for slope that emerged during a design experiment in a high-school Algebra I classroom. In the design experiment, students explored situations related to making predictions. As students engaged with these situations, they reinvented and made-meaningful multiple sub-constructs of slope. I show that this process involved the assemblage and coordination of mathematical artifacts, and I introduce the notion of a *cascade of artifacts* to describe this process. I suggest that artifacts are inextricably bound with activity, and I discuss the nature of the classroom activities that promoted the development of the cascade of artifacts.

A robust understanding of slope is vital for success in secondary and post-secondary mathematics (Thompson, 1994a). However, student understanding of slope is often formulaic and underdeveloped (Stump, 2001). In part this is because slope is a complicated concept, with multiple sub-constructs including "rate of change," "physical property" (steepness), "geometric ratio" (rise over run), "algebraic ratio" (change in y over change in x), and "parametric coefficient" (the a in the equation, $y = ax + b$) (Stump, 1999).

While there has been extensive research on how students come to understand rate of change (Confrey & Smith, 1994; Cramer, Bezuk, & Behr, 1989; Karplus, Pulos, & Stage, 1983; Nemirovsky, 1996; Nunes, Desli, & Bell, 2003; Thompson, 1994b; Tierney & Monk, 2007; Yerushalmy, 1997), comparatively little research has been conducted on how students learn the remaining sub-constructs—especially the connections between the sub-constructs. To investigate how students learn multiple sub-construct of slope, we[1] conducted a design experiment in a high school Algebra I classroom. In this paper, I will discuss the findings from this design experiment. I will show that learning emerged as a *cascade of artifacts*, which I describe as part of a *local instructional theory* (Gravemeijer, 1999, 2004).

The paper is organized as follows: In the next section I present a conceptual framework that explains how artifacts are central to learning. In section 2 I discuss the research and analysis methods. In section 3 I discuss our initial design, including the prior work on slope that informed the design. In section 4 I present the local instructional theory, including the cascade of artifacts. Finally I conclude with implications and directions for future work.

---

[1] In this paper I occasionally use the first-person plural. This is because the design and implementation of the design experiment was conducted by a research team, whereas I conducted the bulk of the analysis and produced this paper. Thus, when I report on a collaborative effort, I use the plural. When I report on my own effort, I use the singular.

Conceptual framework: How artifacts play a central role in learning

In this section I focus on three key concepts, *culture*, *mediation,* and *objectification*, to examine how artifacts play a central role in learning.

I take a process and product view towards culture. Cultural *processes* are those that "accumulate partial solutions to frequently encountered problems" (Hutchins, 1995, pp. 354–355). The residua of these processes – the partial solutions themselves – exist in material and ideal form as cultural *artifacts*. For example, a function table is a common cultural artifact in secondary mathematics. It is a partial solution to the frequently encountered problem of working with two quantities that are in a functional relationship. Function tables have an ideal form, but they are made material in use, often through inscription. That is, the *notion* of a function table is ideal. An inscription—the written manifestation of a function table—is material.

Artifacts, such as function tables, serve to propagate the achievements of past generations into the present, and the set of these artifacts constitutes culture-as-product: "the species-specific medium of human life" (Cole, 2010, p. 462). Notice in this quote that Cole refers to the "medium" constituted by culture. Human actions take place in a cultural milieu and as such they are *mediated* by culture. What this means is that human actions "involve not a direct action on the world but an indirect action, one that takes a bit of material matter used previously and incorporates it as an aspect of action" (Cole & Wertsch, 1996, p. 252). Here, the "bit of material matter" is a cultural artifact. Mediating artifacts do more than simply facilitate or amplify an action that would otherwise exist. Rather, they enable new forms of human actions. Culture, then, is present twice in any human activity. As a process, it is "running" in the background, collecting partial solutions to frequently encountered problems. As a product it is literally "in the middle" (Cole, 1996, p. 116) as artifacts mediate the activity.

So how does this relate to mathematics? Following Freudenthal (1971, 1973, 1991), I define mathematics as the *human activity of structuring the world*. As with any human activity, culture is present twice: as a process and as a product. As a process, culture collects partial solutions to problems that are frequently encountered in structuring the world. The collection of these partial solutions is itself a structure: mathematics as product. Thus, mathematics-as-product is itself an artifact. Using Wartofsky's (1979) hierarchy, mathematics-as-product is a tertiary artifact, one that has been "abstracted from [its] direct representational function" (p. 209), such that it constitutes another world—related to the physical world but not bound to it. As people interact with this artifact it becomes part of their realities. This was Hans Freudenthal's key insight when he claimed that mathematical activity should "start and stay within reality" (Freudenthal, 1987). This was not a call to limit mathematics to the "real world." Rather, it was a call to expand students' "real world" to include mathematics. Learning mathematics is a reality-expanding process.

So far in this discussion I have given the impression that artifacts are purposely-designed objects. This is generally true. Artifacts are imbued with history and this history is manifested as affordances and constraints that shape activity. However, artifacts are not deterministic and they can often afford many types of actions. This is the key insight exploited by Vygotsky and colleagues in their *double stimulation* experiments (Engeström, 2007; Vygotsky, 1978). The method involves giving a subject (in experimental conditions) a task to solve that is “beyond his [*sic*] present capabilities” (Vygotsky, 1978, p. 74). The task is the first stimulus. The experimental setting also contains a neutral object—a second stimulus—which the subject often incorporates into the task:

> [F]requently we are able to observe how the neutral stimulus is drawn into the situation and takes on the function of a sign. Thus, the child actively incorporates these neutral objects into the task of problem solving. We might say that when difficulties arise, neutral stimuli take on the function of a sign and from that point on the operation's structure assumes an essentially different character (Vygotsky, 1978, p. 74).

Thus the second (neutral) stimulus becomes a mediating artifact as the subject pours meaning into the artifact and then uses the artifact to accomplish the task. This is an apt description of my definition of learning in mathematics. More specifically, I define learning as a process of *reinvention* (Freudenthal, 1973, 1991; Gravemeijer, 1999) and *objectification* (Radford, 2008a) in which people endow mathematical artifacts with meaning, incorporating the artifacts into their reality in the process. Internal cognitive processes are surely implicated in this process, but I want to be clear that I am not defining learning as a purely cognitive phenomenon. Mathematical artifacts are cultural not cognitive. Thus learning mathematics does not happen exclusively “in the head” but rather in the cultural world, and the result is not simply a mental structure but rather an expanded reality, composed of mathematical artifacts that are saturated with meaning.

To explain how artifacts are made meaningful, I draw on Latour's (1990) notion of a *cascade of inscriptions*. Recall that above I described how mathematical artifacts are often made material through inscription. For Latour, this material form is the source of the artifact's power. Inscriptions are simultaneously immutable and mobile. They can be brought into coordination with each other and compared, contrasted, and co-manipulated. In other words, the inscriptions themselves can be operated on, resulting in new inscriptions that contain within them the inscriptions that came before them. The process continues, creating a cascade of inscriptions, each containing within it more and more referents. Thus an inscription doesn't just emerge out of thin air, it is built out of—and therefore derives its initial meaning from—other inscriptions. In this paper I will describe how the same process applies to artifacts related to slope. For example, I will describe how students reinvented the geometric ratio ("rise over run") by assembling other artifacts such as number lines, coordinate graphs, and the algebraic ratio ($\frac{y_2-y1}{x_2-x_1}$), and bringing these artifacts into coordination. Because I'm talking about artifacts rather than inscriptions, I describe this process as a "cascade of artifacts." The creation of this cascade is itself mathematical activity: it is the process of structuring the world of mathematical artifacts (sometimes referred to as *vertical mathematization*, Treffers, 1987).

Thus, one way that artifacts take-on meaning is though the artifacts from which they are created. However, once created, artifacts can afford many meanings. The way that they acquire particular meanings is through a social process in which perception is disciplined (Stevens & Hall, 1998), so that particular features and affordances of artifacts become salient. In this way, people learn to "see" artifacts in particular ways (Radford, 2002; cf. Wittgenstein, 1958). This, in turn, happens though the use of *focusing phenomena:*

"regularities in the ways that teachers, students, artifacts, and curricular materials act together to direct attention toward certain mathematical properties over others" (Lobato, Ellis, & Munoz, 2003, p. 1). Often this is accomplished through discourse and gesture (Gee, 2011; Hutchins & Palen, 1997; Radford, 2008b; Streeck, 2009). Taken together, this conceptual framework suggests particular research and analysis methods, which I describe below.

Research and analysis methods

The goal of our research was to develop a *local instruction theory* for slope. Local instructional theories (LITs) are theory-guided frameworks for the design of instructional sequences for a particular topic in mathematics. The word "local" is used to denote the locality of the theory within a single topic in mathematics and to distinguish an LIT from a larger theory of learning. LITs are composed of three parts: (1) a description of how learning happens over time, (2) principles that guide the design of activities that support this learning, and (3) the rationale for how the activities support learning (Gravemeijer, 1999, 2004).

Local instructional theories are developed using design research. This a cyclical method composed of macro-cycles and micro-cycles. At a macro-level there are three phases: preparation, implementation, and retrospective analysis (Gravemeijer & Cobb, 2006; Gravemeijer, 1994, 2004). In this paper, I report on one macro cycle. In the preparation phase we drew on design principles from Realistic Mathematics Education (RME) and literature in mathematics education to create a *conjectured local instructional theory*. In the implementation phase, we implemented the conjectured LIT with students.

The conjectured LIT served as a guide for classroom activities, but it was not "frozen," to be implemented rigidly. Instead, as students engaged with our planned activities, we conducted analysis in-situ—both during and immediately after the activities—and revised and adapted subsequent activities based on this analysis. Thus, the actual path of learning emerged as the result of these cyclic "micro-cycles" of planning, implementation and analysis. In this way the conjectured LIT fed into, but did not dictate, the activities and path of learning that was realized in the classroom. In the same way, the insights that we gained during these micro-cycles fed back into the LIT, which we modified throughout the implementation phase.

At the end of the implementation phase, we were left with a copious amount of data and the task to develop a well-warranted LIT that was consistent with the previous two phases (Gravemeijer & Cobb, 2006). This happened in the retrospective analysis phase. Below I describe the setting for the experiment, the types of data we collected, and the methods that I used to analyze the data in the retrospective analysis phase.

Setting and data collection

We implemented the conjectured LIT in two sections of high-school Algebra I. I was the teacher for both sections (not just for the experiment, but for the entire year). Two other members of the research group served as observers. They captured field notes, and talked with students. The three of us met after each class session to discuss our perceptions of the class, update our conjectured LIT, and make plans for the upcoming class.

The school was located in a suburban area, and served a predominantly white (approximately 60%) and Latino (approximately 30%) population. These figures are based on publicly available data shared on the school's website. We did not have permission to access student demographic information, thus I cannot discuss the specific demographics of the class. However, the school is not tracked and Algebra I is required for graduation, so the makeup of the class was generally consistent with the school as a whole.

The experiment took place in the beginning of the second semester of school. Thus the classes had developed some general routines and norms. In general, there were three forms of activity in the class. (1) Students solved problems in four-person teams that were mostly the same for the entire year. (2) After team-based problem solving, students shared their results in a "math congress" (Fosnot & Dolk, 2001a, 2001b, 2002; Fosnot & Jacob, 2010). (3) Following the math congress, I conducted large-group discussions in which we summarized big results.

As a research teams, we collected student work, observer field notes, and audio and video recordings of full-class and small-group work.

Analysis methods

Because my methods of analysis are connected to my conceptual framework, I begin this section by re-introducing a few important concepts from the conceptual framework. As discussed above, I'm interested in how mathematical artifacts are invented and become meaningful in the classroom. I defined mathematical artifacts as cultural artifacts, and I explained culture as both a process and a product. Cultural processes collect "partial solutions to frequently encountered problems" (Hutchins, 1995, pp. 354–355). Cultural products (that is, artifacts) are the residua of these processes, the partial solutions themselves, which are propagated through time to become resources for a social group in the present. Artifacts then, are inexorably bound to social activity. Artifacts emerge from activity and serve as resources for future activity. In this way activity is constituted through artifacts and so too are artifacts constituted though activity. Indeed it is only through activity that an object emerges as a "partial solution to frequently encountered problems," and it is only through continued activity that a "partial solution" becomes an artifact. Clearly then, for those who wish to study artifacts, the unit of analysis is activity, which I define as:

> [T]he mediated actions and interconnected sequences of actions (i.e. operations) that individuals carry out in the attainment of a goal. This sense of activity, better captured by the German term *Tätigkeit* (as something related to the creative transformation and understanding of reality), [entails a…] fundamental epistemological claim according to which, in the course of the activity, individuals relate not only to the world of objects (the subject-object plane) but also to other individuals (the subject-subject plane or plane of social interaction) and acquire, in the joint pursuit of the goal and in the social use of signs and tools, human experience (Radford, Bardini, & Sabena, 2007, p. 512)

Notice in the above definition that activity is inherently social and interactive. Individuals-with-tools may engage in *actions* (Wertsch, 1998), but these actions can only be understood insofar as they come together in a constellation called "activity" (Engeström, 1987; Leont'ev, 1981). For example, a student using a calculator to divide 115 by 5 constitutes an action, which can only be understood in the larger context of a group's joint work in solving a problem for which the mathematical operation of 115/5 is meaningful. Even this larger unit of group work is not activity as defined above. Activity is broader: "doing mathematics" is activity. Solving a school problem is (at best) a moment of this activity (Radford et al., 2007). These moments are my units of analysis, and my task is to understand how artifacts emerge and become meaningful within and across these moments. That is, my task is to capture culture processes, by examining cultural products and how they are used.

To do so, I used a cyclical process of analysis. In the first phase, my goal was to determine the artifacts that constitute the LIT and the order in which these artifacts emerged. In this round, I watched video data collected in the classroom sequentially, identifying moments of activity in which artifacts were reinvented or objectified. In the second round, my goal was to understand how these artifacts were reinvented and objectified. I therefore focused on the key segments that I identified in the first round. Because objectification is often a sensuous process that is distributed across people and inscriptions via talk and gesture, I transcribed these segments for talk and gesture and coordinated the transcriptions with inscriptions produced by students. Using these data sources, I analyzed how artifacts were reinvented and objectified in interactive moments of activity.

To analyze interaction I drew on techniques that can broadly be categorized as discourse analysis (Gee, 2011; Heritage & Clayman, 2010; Potter, 1996), which I define as including both talk and gesture (Streeck, Goodwin, & LeBaron, 2011; Streeck, 2009). In analyzing interaction, I was guided by the notion that discourse is not just communication about action, it is itself action. That is, discourse communicates, but it also *does* (Gee, 2011; Jaworski & Coupand, 2006). A key question that I asked during the analysis is, “what work is *x* doing?” where *x* might refer to a single word or phrase, a turn at talk, a gesture, an artifact, a math problem, or a feature of the situation (e.g., the imaginary context within which a particular problem is set). Evidence for this work is found in subsequent interaction.

While the above suggests a positivist notion that theory emerges from neutral observation of neutral data, I want to be clear that both the data and my observations are theory-laden. The data are theory-laden because they come from a sequence of activities that were designed based on a conjectured LIT that was informed by theory. The observations are theory-laden because they were informed by both the conjectured LIT and my conceptual framework. For example, I made a choice to focus my observations on artifacts. This choice comes from my conceptual framework, and it colors every part of my data analysis. A different analyst could make a different choice and she would see things in the data that I do not. Furthermore, I was the teacher of the course. I have fond memories of the class and of my students, and it would be folly to imagine that these memories do not enter into my analysis. The validity of my results, then, is not located in the neutrality of the data or analysis, but rather in the reasonableness and justifiability of my claims in light of my conceptual framework (Gravemeijer & Cobb, 2006).

The design of the conjectured LIT

In designing the conjectured LIT, we drew on (a) the theories of learning described in the conceptual framework, (b) design heuristics from Realistic Mathematics Education (RME), and (c) research in math education related to slope. In this section, I discuss the latter two categories, relating them back to the theories of learning described in the conceptual framework. I then describe our conjectured LIT.

RME design heuristics

We drew on three heuristics from RME: emergent modeling, guided reinvention, and didactical phenomenology. In the conceptual framework I explained that learning is a process of reinvention and objectification, through which students invent mathematical artifacts and make them meaningful. From a design perspective, the "final" artifacts are known. In our case these artifacts are the five sub-constructs of slope, discussed in the literature review below. Our task as designers was to create a sequence of activities in which students were *guided to reinvent* these artifacts as partial solutions to frequently encountered problems. This reinvention happens through a process of *emergent modeling*. The general idea is that students create mathematical artifacts and make them meaningful as they engage in problem solving. Like Vygotsky's double stimulation experiments, the activities are structured so that they are just "beyond [the students'] present capabilities." Students solve these problems by creating new artifacts and incorporating the artifacts into their problem solving. The task for the designer is to sequence the problems so that artifacts emerge in a meaningful sequence: at any given time a particular artifact signifies the artifacts that came before it, and later it will be signified by a more general artifact. In this way, the "final" artifacts (the five sub-constructs of slope in our case) emerge as just one step in this cascade (Gravemeijer, 1999; Latour, 1990). For example, as I will explain later in this paper, Cartesian graphs originally were signs for our students, signifying tables of coordinate pairs. Later, Cartesian graphs began to acquire new meaning, and were themselves signified by the construct of "rise over run."

In designing activities, we were guided by the RME principle of *didactical phenomenology*. The idea is that students should be presented with rich contexts that (a) “are begging to be organized” (Gravemeijer & Terwel, 2000, p. 787) and (b) can be organized by the artifact that is meant to be invented. In addition, we were guided by the literature from math education related to slope, as I describe below.

Review of literature in math education related to slope

Slope is composed of seven sub-constructs: (1) rate of change, (2) physical property (steepness), (3) geometric ratio (rise over run), (4) algebraic ratio ($\frac{y_2-y_1}{x_2-x_1}$), (5) parametric coefficient (the a in the equation, $y = ax + b$), (6) trigonometric ratio (the tangent of the angle that a graphed line makes with the x-axis), and (7) derivative of a function (Stump, 1999). We focused our study on the first five of these, as only these five were part of the Algebra I curriculum of the school (the latter two sub-constructs were introduced in later courses). Of the first five sub-constructs, *rate of change* has received the most scrutiny from researchers. In what follows, I will summarize some of this large literature as it relates to our study. From there I will summarize work that has explored the other sub-constructs.

Much of the work on rate has explored how students come to understand rate through covariation (Confrey & Smith, 1994; Lobato et al., 2003; Thompson, 1994b). Covariation can be constituted through tables. Working with tables, students coordinate changes in one variable with changes in another by moving up or down a well-ordered table (Confrey & Smith, 1994). This representation can make salient the changes from row to row, but it can also afford outcomes in which covariation is cast as differences rather then a ratio (Lobato et al., 2003; Schliemann & Carraher, 2000). This is mediated by particular focusing phenomena, including the form of the representation and the language used to describe changes within the representation. Specifically, well-ordered tables where the independent variable increases by one in each subsequent row can lead to “goes up by” language (i.e., “it goes up by three” as a way to describe the covariation present in a well-ordered table where the independent variable increases by 1 and the dependent variable increases by 3 in each subsequent row). When students and teachers use this language, they are attending to only one of the two variables that are changing, and thus covariation is cast in terms of differences rather than a ratio (Lobato et al., 2003).

Others have explored how students come to understand rates as measures of intensive quantities, for example of steepness, speed, or intensity of taste (Karplus et al., 1983; Lobato & Thanheiser, 2002; Nunes et al., 2003; Simon & Blume, 1994). From a covariational perspective, rates describe dynamic phenomena: two quantities “accrue simultaneously and continuously, and accruals of quantities stand in the same proportional relationship with their respective total accumulations” (Thompson, 1994a, p. 232). In contrast, from a measurement perspective a rate is a relatively static object. For example, when one creates a measure of speed in order to answer a question about “which girl was running faster?” (Karplus et al., 1983, p. 222), speed is reduced to a single number; dynamic notions of simultaneous accruals over time are not at the fore.

Both interpretations of rate are important, and indeed, they can inform each other. For example, Lobato et al. (2003) speculated that understanding rates as measures of intensive quantities would help students understand rates as covariation. Specifically, they speculated that students who understand rates as measures would be less likely to cast covariation in terms of differences (rather than ratios). Thus, these authors recommend that students should have experiences in which they create rates as measures of intensive quantities using division, and that students should use the language of intensive quantities to describe these rates. As I discuss in the next sections, this recommendation played a large role in the design and execution of our study.

Much of the research on the other sub-constructs of slope involves linking two sub-constructs together (Herbert & Pierce, 2005; Herbert, 2008; Lobato & Ellis, 2002; Tierney & Monk, 2007). Often, this involves multiple functions in a single representation or multiple representations of a single function. As an example of the former, graphs that show multiple linear functions at once can help constitute the connection between slope-as-steepness and slope-as-rate (Tierney & Monk, 2007).

With respect to the latter, many authors have argued that different representations (e.g., tables, graphs, and algebraic equations) make salient different aspects of functions, and this is certainly the case for slope—especially because certain sub-constructs are only manifest in a particular representation (for example, slope-as-parametric-coefficient is only manifest in algebraic equations, whereas slope-as-geometric-ratio is only manifest in graphs in the Cartesian plane). However, simply providing multiple representations is not sufficient to link interpretations of slope across representations. Instead, "explicit connections between [representations] are required to enable students to transfer understandings of rate from one representation to another" (Herbert, 2008, p. 34).

This is consistent with the concepts that I outlined earlier. The design principle of emergent modeling provides a heuristic for designing activities such that the meaning of one artifact is built on of the meanings of others. In this way, the "explicit connections between [artifacts]" is inherent in the process through which students make artifacts meaningful.

The conjectured LIT

Drawing on the RME design heuristic of didactical phenomenology, we asked ourselves, "what sort of contexts are (A) begging to be organized, and (B) can be organized by the five sub-constructs of slope?" The canonical context for slope is steepness. However, we rejected this context because it seemed to fail both criteria. With respect to criterion A, we did not feel that finding steepness would be motivating enough to sustain an entire unit of study. With respect to criterion (B) it was not clear to us how steepness could be used to organize the five sub-constructs of slope. Steepness is clearly connected to the physical property (it *is* the physical property). And it's not too big of a leap to see how one measures steepness using the geometric ratio. Beyond these geometric sub-constructs however, steepness falls short. For example, consider the parametric coefficient. In the equation $y = ax + b$, why would someone multiply steepness by x? What problem is this solving? While it is certainly possible to find a scenario where this might be reasonable, it seemed to us to be implausible at best.

Another option is a purely formal treatment, predicated on the connection between coordinate graphs and algebraic equations. This too seemed to fail criterion (A), in that for students, formal mathematics is not something that is begging to be organized.

Ultimately, we decided to focus the unit on making predictions using mathematics. This seemed to meet both criteria. With respect to criterion A, making predictions about the future seemed to us to be a fairly motivating context. With respect to criterion B, predictions are not tied to a single representation or a single sub-construct of slope. In addition, organizing the unit around predictions allowed us to center the unit on the key concept of rate of change (as recommended by researchers and professional organizations

[Confrey & Smith, 1994; NCTM, 2000; NGA, 2010; Stroup, 2002; Stump, 2001; Thompson, 1994a]), and allowed us to build on students' vast repertoires of proportional reasoning experience.

We next considered the principles of guided reinvention and emergent modeling to design the following sequence of activities:

1. In situations involving predictions, students reinvent and objectify rate of change from a covariation perspective and a measurement perspective (based on the work of Confrey & Smith, 1994; Lobato et al., 2003; Lobato & Thanheiser, 2002). In addition, students reinvent and objectify the parametric coefficient as a way of using rates of change to make predictions.
2. By assembling and coordinating rates of change, proportional reasoning, and multiple representations of functions, students reinvent and objectify the algebraic ratio and geometric ratio as ways to calculate a rate of change (this extends the work of Brenner et al., 1997).
3. By examining the relationship between the rate of change and the shape of a linear graph, students reinvent the physical property as a measure of steepness (based on the work of Lobato & Thanheiser, 2002 and Tierney & Monk, 2007).

Units on slope often start with graphs of linear functions in a coordinate plane. However, we delayed the introduction of graphs for two reasons. First, we felt that graphs are often poor representations for making predictions. Second, we knew that many students had learned slope as a rule for finding steepness of graphed lines in middle school.

We wanted students to reinvent slope in a new way, thus we wanted to delay this familiar representation.

A local instructional theory for slope

In this section, I discuss the local instructional theory that emerged as we conducted the design experiment. Recall that LITs are composed of three parts: (1) a description of how learning happens over time, (2) principles that guide the design of activities that support this learning, and (3) the rationale for how the activities support learning (Gravemeijer, 1999, 2004).

I present the description of how learning happened over time in the form of a *cascade of artifacts* (Figure 1), which shows how new artifacts were reinvented and objectified as assemblages and combinations of existing artifacts. I represent the cascade as a directed graph. As such, it shares some surface similarities with “cognitive models” (Gierl, Wang, & Zhou, 2008), however the cascade is not a representation of an internal cognitive structure. Instead, it is a representation of how artifacts (which, recall, exist in the cultural world, not in the head) inform and are informed by other artifacts. That said, the cascade is not a representation of artifacts from a disciplinary perspective. Rather, it is a representation of artifacts from a *learning perspective*. Thus it is not built from a disciplinary (or expert) perspective but rather it is shows how students structured their mathematical world by assembling and coordinating mathematical artifacts.

The cascade is organized such that the vertical dimension is somewhat meaningful but the horizontal dimension is not. In general, the cascade flows downward with artifacts that are higher in the cascade contributing to those that are lower. Thus the cascade can be "read" from top to bottom, as shown by the downward facing arrows in Figure 1. However, artifacts don't just "push down," they also push up and push laterally, as shown by the many double-facing arrows in Figure 1. Thus objectification is a synchronistic and symmetric process in which artifacts objectify and are objectified by each other, all at the same time. While there is a general downward push in the cascade, it would be a mistake to conclude that any given artifact is a pre-requisite for another.

The cascade was constituted in six ordered stages. This is shown on the bottom of Figure 1, where the artifacts that are invented in each stage are highlighted in red, and the artifacts that are assembled and coordinated in the stage are shown in yellow. In introducing the notion of *order*, I am introducing a tension. On the one hand, there is a purposeful order to the stages. On the other hand, because artifacts push up and sideways, later stages influence earlier stages just as earlier stages influence later stages. Rather than minimize this tension I embrace it and suggest that the best way to conceptualize a given stage is to think of it as informing, simultaneously, the present, the future, and the past.

The six stages of the LIT, along with the activities through which the stages are constituted, are summarized in Table 1. In the remainder of this section I elaborate each stage, focusing on how activities and artifacts support reinvention and objectification at that stage. For space purposes, I only offer a detailed analysis of Stage 3 (the first "official" stage in the design experiment). The remaining stages are summarized with supporting evidence.

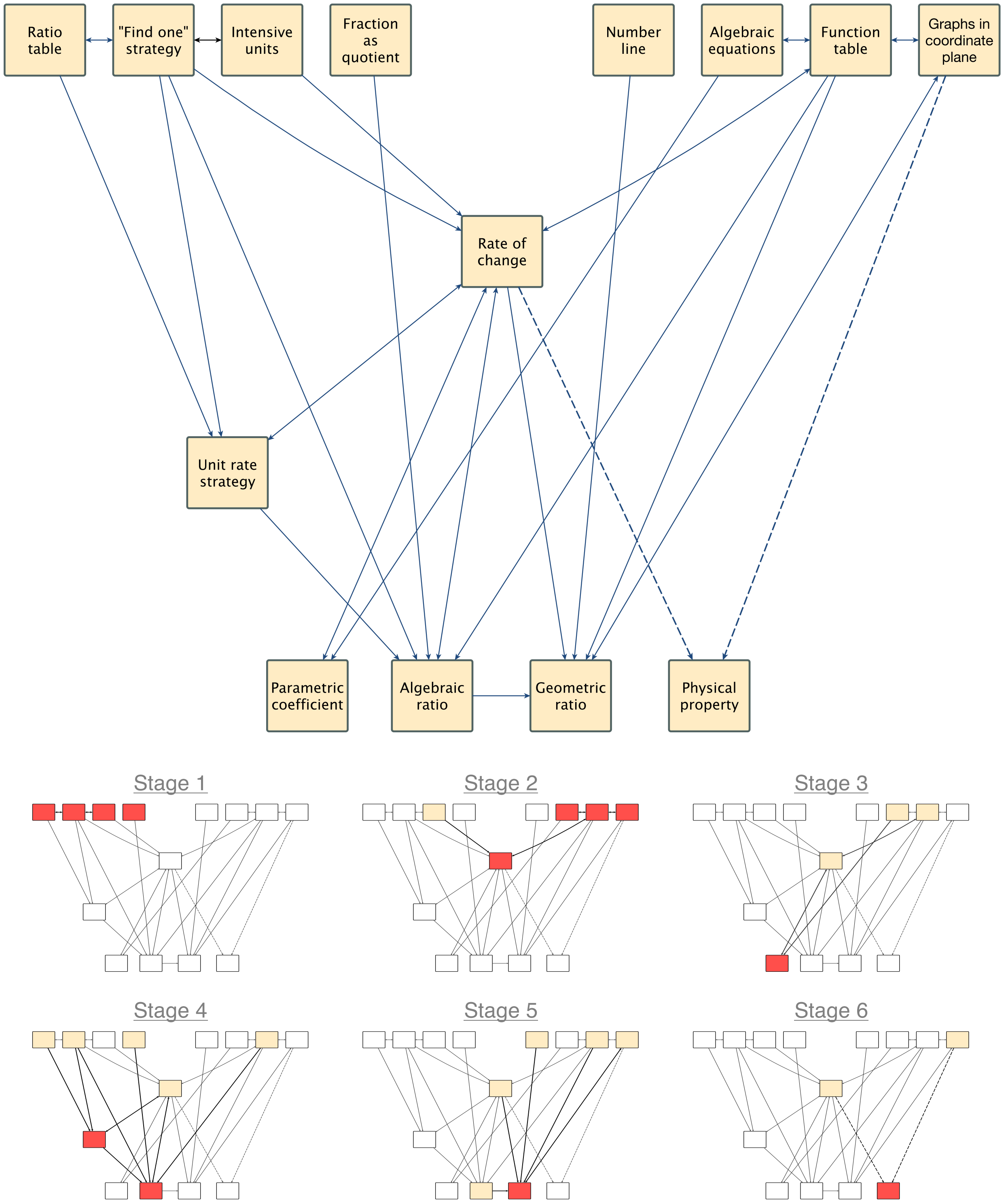
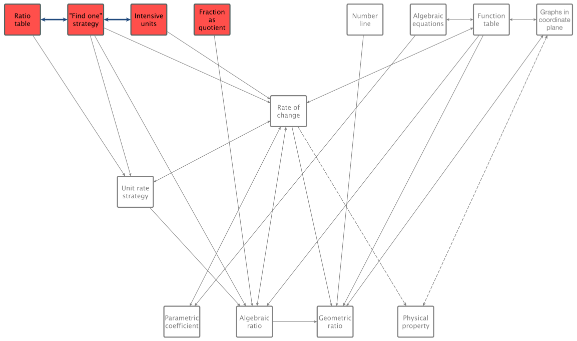
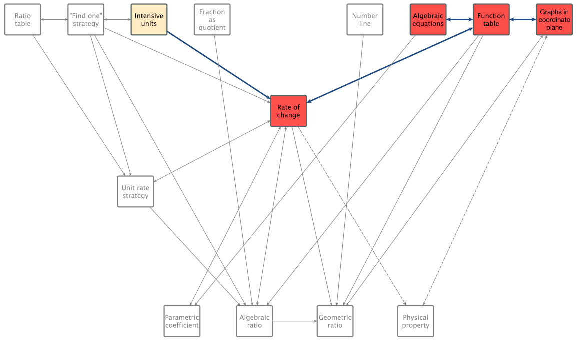
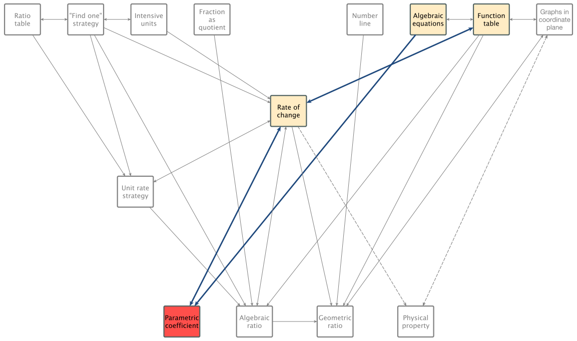
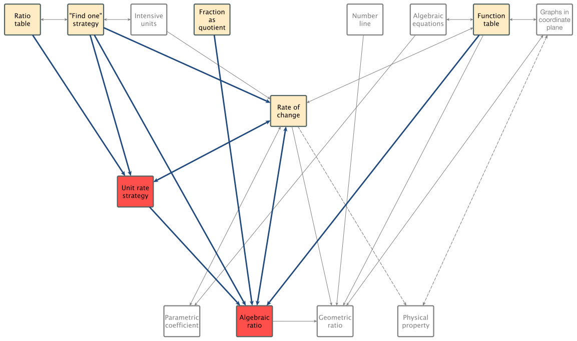
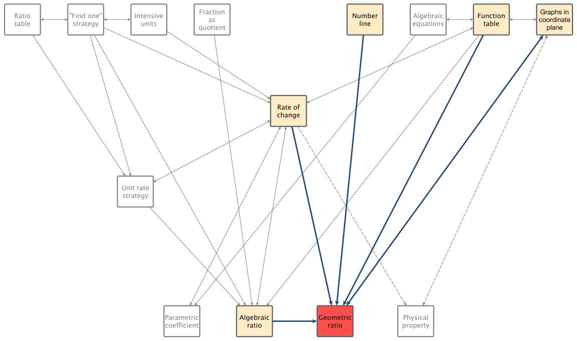
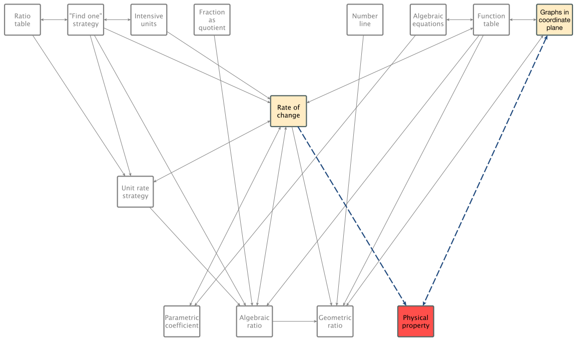

Ratio table
"Find one" strategy
Intensive units
Fraction as quotient
Number line
Algebraic equations
Function table
Graphs in coordinate plane
Rate of change
Unit rate strategy
Parametric coefficient
Algebraic ratio
Geometric ratio
Physical property
Stage 1
Stage 2
Stage 3
Stage 4
Stage 5
Stage 6

Figure 1. Learning progressed in a cascade of artifacts (top), which was constituted in six stages (bottom). The dotted lines represent conjectured relationships. In the bottom figures, the artifacts that were assembled and coordinated in each stage are shown in yellow, and those that were reinvented at each stage are shown in red.

Table 1. Overview of the LIT

Stage		Artifacts	Characteristics of tasks
1	 Ratio table "Find one" strategy Intensive units Fraction as quotient Number line Algebraic equations Function table Graphs in coordinate plane Rate of change Unit rate strategy Parametric coefficient Algebraic ratio Geometric ratio Physical property	Reinvented & objectified: • Ratio table • "find one" strategy • Intensive units • Fraction-as-quotient	Tasks that involve the activity of partitive division, including: • finding fair shares • finding unit values
2	 Ratio table "Find one" strategy Intensive units Fraction as quotient Number line Algebraic equations Function table Graphs in coordinate plane Rate of change Unit rate strategy Parametric coefficient Algebraic ratio Geometric ratio Physical property	Assembled and coordinated: • Intensive units Reinvented & objectified: • Algebraic equations • Function tables • Graphs in coord. plane • Rate of change	Tasks that involve: • Finding and continuing patterns in geometric figures and tables of values, where there is a "starting value" and the independent variable increases by 1 • Converting between multiple representations of functions (focusing on table rows and points in the plane as *solutions* to two-variable equations)
3	 Ratio table "Find one" strategy Intensive units Fraction as quotient Number line Algebraic equations Function table Graphs in coordinate plane Rate of change Unit rate strategy Parametric coefficient Algebraic ratio Geometric ratio Physical property	Assembled and coordinated: • Algebraic equations • Rate of change Reinvented & objectified: • Parametric coefficient Objectified • Rate of change • Function tables	Making predictions in linear situations, given: • The rate of change and starting value • Multiple data points (e.g., in a table), where the independent variable increases by one.

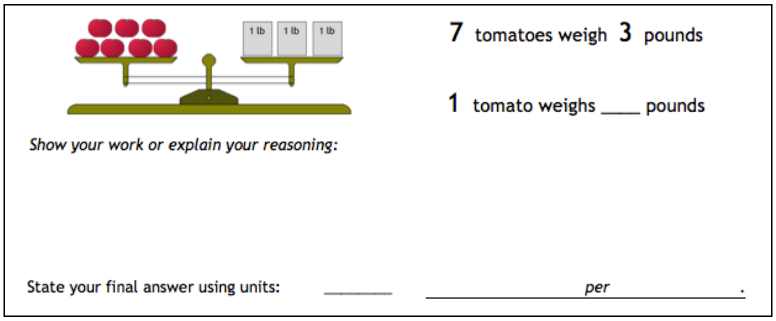
Stage	Artifacts	Characteristics of tasks
4  Ratio table "Find one" strategy Intensive units Fraction as quotient Number line Algebraic equations Function table Graphs in coordinate plane Rate of change Unit rate strategy Parametric coefficient Algebraic ratio Geometric ratio Physical property	Assembled and coordinated: • "Find one" strategy • Ratio table • Fraction as quotient • Function tables • Rate of change Reinvented & objectified: • Unit rate strategy • Algebraic ratio Objectified • Rate of change	Make predictions in linear situations given: • A single data point, for situations where the values of the variables are proportional • Two data points, for situations where there is a starting value. Problem contexts should be chosen to make clear the distinction between changes and values.
5  Ratio table "Find one" strategy Intensive units Fraction as quotient Number line Algebraic equations Function table Graphs in coordinate plane Rate of change Unit rate strategy Parametric coefficient Algebraic ratio Geometric ratio Physical property	Assembled and coordinated: • Number line • Graphs in coord. plane • Rate of change • Algebraic ratio Reinvented & objectified: • Geometric ratio Objectified: • Graphs in coord. plane	• Show change on number-line diagrams. • Make predictions in linear situations where there is a starting value, given a graph of a function in a coordinate plane.
6  Ratio table "Find one" strategy Intensive units Fraction as quotient Number line Algebraic equations Function table Graphs in coordinate plane Rate of change Unit rate strategy Parametric coefficient Algebraic ratio Geometric ratio Physical property	Assembled and coordinated: • Rate of change • Graphs in coord. plane Reinvented & objectified • Physical property	• Compare rates given two intersecting linear functions graphed in a coordinate plane. • Measure and compare the steepness of objects

Stages 1 and 2

Stages 1 and 2 occurred before our design experiment started. I discuss them here because the artifacts that were reinvented and objectified in these sessions formed the basis for the artifacts in the design experiment. I don't have access to student-level data to drawn on for analysis of these stages. However, because I was the teacher of the course I do have access to my course records, and I draw on these records to explicate these stages.

Stage 1

Stage 1 took place in the beginning of the year during a short unit on fractions-as-quotients (see Peck & Matassa, in review, for a detailed summary of a similar unit). In this stage, students solved problems that involved finding unit values given a many-to-many relationship (the sort of problems that a mathematician might classify as involving partitive division). For example, students solved fair-sharing problems using equipartitioning (Empson, 1999; Streefland, 1993; Wilson, Edgington, Nguyen, Pescosolido, & Confrey, 2011), as well as other problems such as finding the weight of a single tomato given the weight of multiple tomatoes (see Figure 2). In situations like these, there are two ways to conceptualize the "final answer." A *many-as-one* conceptualization attends to only one of the dimensions in the many-to-many relationship, while a *many-to-one* conceptualization attends to both dimensions. (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009). In the example shown in Figure 2, a student who used a many-as-one conceptualization would express the answer as "3/7 pounds," whereas a student who used a many-to-one conceptualization would express the answer as "3/7 pounds per tomato." As shown in Figure 2, we encouraged the latter conceptualization. We did so because the many-to-one conceptualization more readily lends itself to the notion of covariance (Confrey et al., 2009). Even though covariance was not a focus of this unit, we knew that it would be a focus of our design experiment. In this way the unit was future-oriented, preparing students for future stages, months in the future, in which covariation would come to the fore.



7 tomatoes weigh *3* pounds

1 tomato weighs ____ pounds

Show your work or explain your reasoning:

State your final answer using units: ________ ____________ *per* ____________.

Figure 2. Example of a “find the unit value” problem

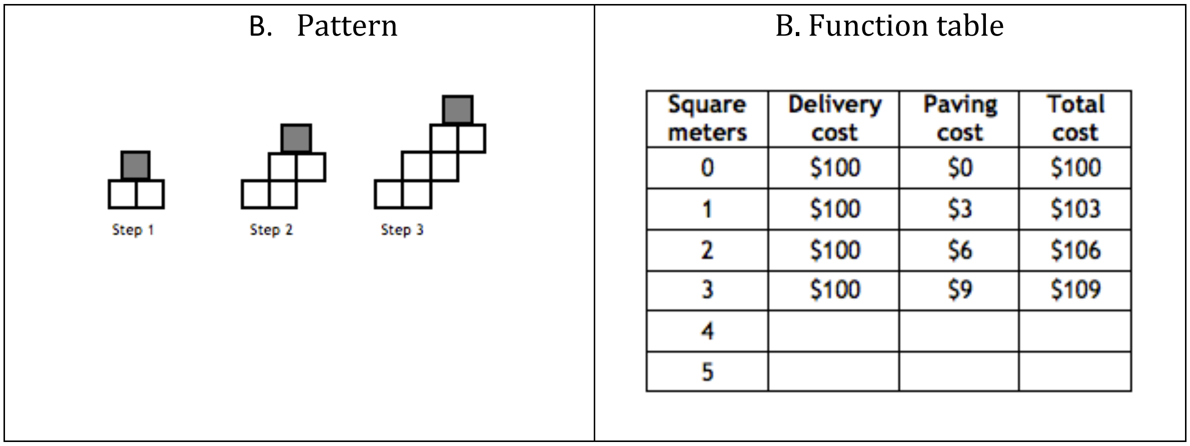
Through these problems, the students invented and/or objectified the following mathematical artifacts, which later served as the foundation for objectifying rates:

1. The fraction-as-quotient sub-construct of rational number (Kieren, 1980), as well as the notion that the “fraction bar” can serve as a division operator. Thus students’ perception was disciplined such that they could see the symbol ¾ as simultaneously the operation of three divided by four, and the numerical result of the operation, three-fourths.
2. The “find-one” strategy. This strategy was invented and named by the class during the course of the unit. It links the division operation to situations in which the goal is to find the value of one object.

3. The ratio table (Middleton & van den Heuvel - Panhuizen, 1995). Notice that the tomato problem in Figure 2 is presented such that within-unit pairs are aligned vertically, and between-unit pairs are aligned horizontally. Students used this structure as a ratio table, and eventually reproduced it as they solved new problems (discussed further in the summary of Stage 4, see, e.g., Figure 6).
4. Intensive units using the word "per" (e.g., pounds per tomato). The word "per" became synonymous with the notion of "one-ness" and, as discussed above, we used intensive units to maintain both of the original dimensions in many-to-one situations (Confrey et al., 2009).

Stage 2

Stage 2 took place during a unit on functions, including vocabulary and concepts (function, independent and dependent variables, inputs and outputs) and common representations (including tables, graphs, equations, words, and "arrow chains", which were models of function machines). In this stage, students engaged in the following activities: (1) finding and continuing patterns in geometric figures and function tables, and (2) representing patterns using multiple representations of functions. Although the main goal of the unit was to introduce the concept of a function and its associated vocabulary and representations, a secondary goal was to discipline students' perception such that they would see how a linear function's output could be composed of a constant part and a changing part. As shown in Figure 3, we incorporated visual focusing phenomena to make these separate components salient (inspired by the use of tables in Brenner et al., 1997). All of the patterns were presented such that the independent variable increased by one for each subsequent iteration. We were mindful of Lobato et al.'s (2003) caution that such well-ordered representations can lead to students' perceptions being disciplined to changes in the dependent variable only (i.e., without considering the simultaneous change in the independent variable). We were therefore very careful to use focusing phenomena that called attention to both variables, such as quantifying covariation using intensive units (e.g., cost per square meter).


B. Pattern

B. Function table

Square meters	Delivery cost	Paving cost	Total cost
0	$100	$0	$100
1	$100	$3	$103
2	$100	$6	$106
3	$100	$9	$109
4			
5			

Figure 3. Two examples of visual focusing phenomena that we used to discipline students' perception to see a linear function's output in terms of a constant part and a changing part.

Through these problems, the students reinvented and/or objectified the following mathematical artifacts:

1. Function tables. This was the primary representation used in the unit. Students used tables to collect numbers extracted from geometric patterns, and tables served as bridges between other representations of functions. For example, students used tables to convert from algebraic equations to graphs by using the table to collect solution pairs for the equation, and then plotting the pairs on a graph.
2. Rate of change. This was the first of the five sub-constructs of slope that students reinvented and objectified. Above I discussed how we used representations to

discipline students' perception to the change in the dependent variable for every unit change in the independent variable. As students called attention this covariation, I defined "rate of change" as "the amount that the output changes by when the input goes up by one" (class records, 11/16/2011). Students used intensive units to quantify rates, and we discussed how rates play a role as an "exchanger" (class records 11/16/2011), converting the independent variable to the dependent variable through multiplication. As shown in Figure 1, and discussed earlier, rate of change is the central artifact in the cascade, and it informed all of the other sub-constructs of slope.

3. Algebraic equations. Equations were used as "rules" (class records 11/04/2011) to find the output for any input in a particular pattern. We discussed how there were infinite solutions to these equations, each representing a different possible input, output pair.
4. Graphs in the coordinate plane. Graphs were objectified as the collection of points that made a particular equation true. We introduced graphs as a visual representation of the set of the infinite solutions to two-variable equations. As discussed above, students created graphs from equations by first finding solution pairs to the equation, and then plotting them. Similarly, equations were used as "point-checkers" (class records 12/06/2011) to determine if a given point would be on a graphed line.

In summary, eight key artifacts, including one sub-construct of slope, were invented and/or objectified in Stages 1 and 2. In both stages, the notion of many-to-one was pervasive. In Stage 1, the entire unit was oriented around "finding one." In Stage 2, students explored patterns in which changes in dependent variables were associated with unit changes in independent variables, leading ultimately to a "many-per-one" definition of rate of change.

Stage 3

Stage 3 marked the beginning of the design experiment. In this stage we introduced the theme of the unit, "using math to make predictions." Students solved two different categories of problems, both of which involved making predictions in linear situations: (1) making predictions where the rate of change and starting value are given; and (2) making predictions where multiple data points are given (e.g., in a table), and the independent variable increases by one (this second category is similar to the problems that students encountered in stage 2, but this time there was an explicit focus on making predictions rather than finding rules).

As students solved these problems, they reinvented the parametric coefficient by coordinating rate of change with algebraic equations. Recall that the parametric coefficient is the a in the linear equation, $y = ax + b$. We wanted to discipline students' perception such that they saw the ax term as one in which the rate is being multiplied by the independent variable to make a prediction. We conjectured two different ways that students might objectify rate in order to see the ax term in this way, both related to the ways that rate of change was objectified in stage 2. One way is to objectify rate as an "exchanger" that works though multiplication to exchange the independent variable for the dependent variable. A second way is to objectify rate as a many-per-one relationship, which can then be accumulated through multiplication. Overwhelmingly, as I discuss below, rate was objectified as the latter.

In the initial activity, students read news articles and blog posts in which authors presented rate made predictions based on those rates. We then discussed how the authors made their predictions, and disciplined students' perception to see that rates were being multiplied by an independent variable to make predictions. For example, in an article about Apple iPhones (Lane, 2008) the author describes that Apple is manufacturing 800,000 iPhones per week, and later suggests that "[a]t the current rate, Apple stands to produce more than 40 million iPhone 3Gs over the course of twelve months." After students read the article we had a discussion about the prediction. In the discussion, students identified the "800,000 iPhones per week" as the rate of change and explained that the author could make the prediction by multiplying 800,000 by 52.

The students were explaining a straightforward multiplication situation, the sort that they had probably seen since early elementary school. We wanted use this simple task as a way to help students see something new in the multiplication, namely the way that rates can be used to make predictions through multiplication. Thus I asked students *why* we would multiply. This interrogative is a request for detailed information, and works to reframe the activity away from the calculation. Randy[2] explained:

[1]: Why multiplication?

1 FAP: Randy why is that [multiplication] going to get us a prediction for the number of iPhones in a year? How does weeks turn into iPhones?

2 Randy: Because for every week you have, you produce a certain amount of iPhones, so if you multiply it by a certain amount of weeks, the amount of iPhones will go up. [The reason-

3 FAP: [For every –

4 Randy: -that might be important is for (investors to know)

---

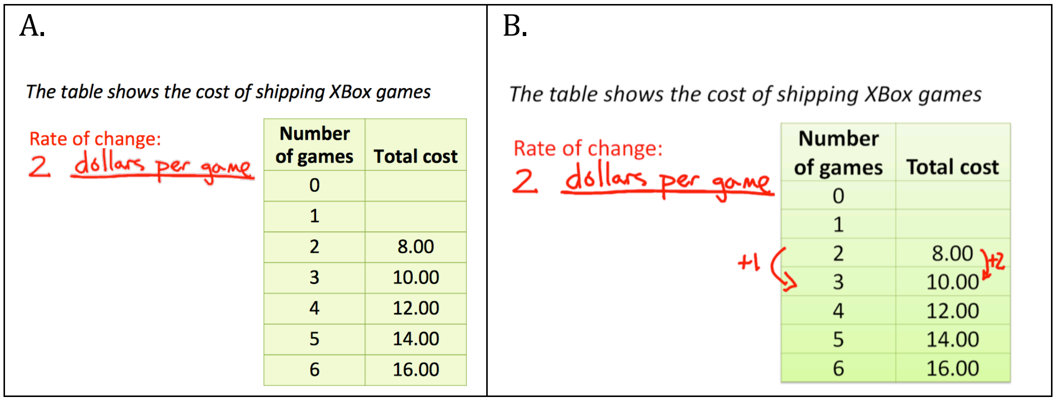
[2] All student names are pseudonyms.

I begin turn 1 with the why interrogative. I ended my turn with a question about how “weeks turn into iPhones.” In doing so, I offered the “exchanger” objectification of rate discussed above. However Randy provides an explanation that puts the many-per-one meaning into rate of change. This can be seen in turn 2. In this turn, the phrase “for every” plays the role of “per”, associating the singular “week” (one) with the plural “iPhones” (many). The determiner “every” makes clear that the week in question is not a one-off event, but rather can be repeated (consider how the meaning would be different if Randy had used a different determiner, say, “the”, instead of “every”).

Having objectified rate as a many-per-one relationship, Randy further objectifies rate when he says: “if you multiply it by a certain amount of weeks, the amount of iPhones will go up.” Here, Randy is doing more than explaining how he sees the multiplication problem. He is objectifying rate—publicly—as a number that can be accumulated through multiplication to make predictions. In subsequent problems, students continued to objectify rates in this way, and coordinated this objectification with algebraic equations to reinvent and objectify the parametric coefficient. For example, when analyzing the table in the “X-box problem” (Figure 4A), students identified the rate using intensive units as “2 dollars per game.” Stacy explained how she saw the rate in the table (Excerpt [2]):

[2]: Seeing rate in a table

1 FAP: Stacy, where do you see that rate of change in the table?
2 Stacy: Um, for every number of games, the total cost goes up by two
3 FAP: Ever::y what about the number of games?
4 Stacy: The, each time the number of games increases by one, the total cost increases by two.
5 FAP: (draws arrows on table, one on each side, pointing from one row to the subsequent row. On the left-side arrow, writes “+1” and on the right-side arrow writes “+2”; See Figure 4)


A.

The table shows the cost of shipping XBox games

Rate of change:
2 dollars per game

Number of games	Total cost
0	
1	
2	8.00
3	10.00
4	12.00
5	14.00
6	16.00

B.

The table shows the cost of shipping XBox games

Rate of change:
2 dollars per game

Number of games	Total cost
0	
1	
2	8.00
3	10.00
4	12.00
5	14.00
6	16.00

Figure 4. Seeing rate of change in a table. (A) shows the table at the beginning of Excerpt [2], and (B) shows the table at the end of Excerpt [2]

In this exchange, Stacy and I work together to objectify both function tables and rate of change. The sequence in turns 2-4 works to objectify rate as many-per-one. In turn 2 Stacy uses similar "for every" language as Randy did in [1]. However, Stacy uses a mass noun: for every *number* of games," thus making it ambiguous whether she is making a many-per-many association or a many-per-one. In turns 3 and 4, she and I clarify that she is making a many-per-one association. In turn 5, I bring this objectification into coordination with the function table, drawing arrows to discipline perception such that rate becomes salient in the movement between table rows. This shows how rate "pushed up" in the cascade to further objectify function tables.

Later, a different student (Melissa) wrote $y = 2x + 4$ as an equation for the table. She explained:

[3]: Seeing rate in an equation

1 Melissa: Okay, um Y is like the final, cost, and,
→ two is the one time fee times how many games you have- or not the one time fee
→ like, how much dollars it is per game, and four is the one time fee.

In this turn, Melissa explains how she sees quantities in the symbols that form the equation. Of interest to us are the two arrowed passages, as it is in these passages that Melissa objectifies the parametric coefficient. In the first arrowed passage Melissa makes a mistake, referring to the 2 as the "one-time fee" She soon initiates a self-repair, but not immediately. The repair comes after she explains that the one-time fee is multiplied by "how many games you have." The timing of this repair suggests that it is brought on by Melissa's recognition that the role she has ascribed to the "2" doesn't fit with its behavior in the equation: a "one-time fee" would not be accumulated through multiplication. The presence and timing of the repair therefore suggests that Melissa has objectified the $2x$ term, and that her explanation does not fit her objectification.

Having initiated the repair, in the second arrowed passage Melissa explains (correctly) that "how many dollars it is per game" is being multiplied by the number of games. In doing so, she draws on the objectification of rate as many-per-one and coordinates this objectification with its role in the algebraic expression. In this way, Melissa assembles and coordinates rate of change with algebraic equations to objectify the parametric coefficient.

The above shows how students assembled rates, function tables, and algebraic equations to reinvent and objectify the parametric coefficient. In addition, the above discussion shows how students continued to objectify rates of change. Students put a "many-per-one" meaning into rate of change, and objectified rates as numbers that can be accumulated through multiplication. As rates became meaningful in this way, they were objectified as a tool to make predictions. In a quick-write at the end of this stage, we asked students, “why are rates of change useful?” As shown in the word cloud in Figure 5, two of the most frequent words in the students’ responses were “predict” and “future.”

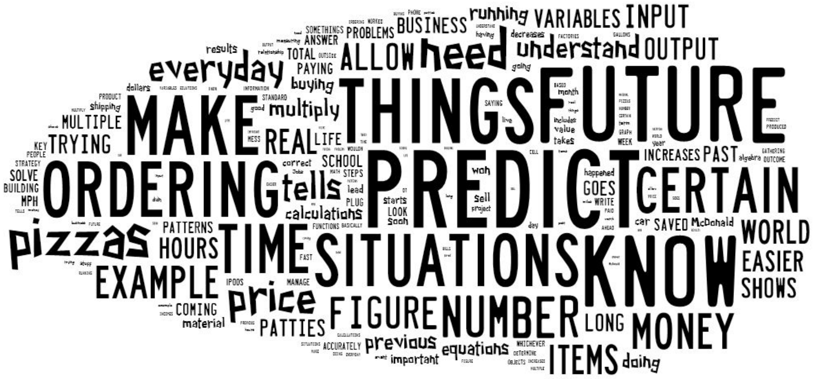

running
VARIABLES
INPUT
BUSINESS
PROBLEMS
SOMETHINGS
ANSWER
results
everyday
ALLOW
need
understand
OUTPUT
TOTAL
PAYING
buying
decreases
dollars
MAKE
THINGS
FUTURE
multiply
shipping
MULTIPLE
TRYING
REAL
LIFE
month
value
takes
includes
KEY
PEOPLE
SOLVE
BUILDING
MPH
ORDERING
tells
PREDICT
CERTAIN
correct
SCHOOL
STEPS
lead
PLUG
calculations
starts
LOOK
soon
sell
happened
GOES
WRITE
INCREASES
PAST
car
SAVED
McDonald
pizzas
PATTERNS
HOURS
TIME
SITUATIONS
KNOW
WORLD
EASIER
SHOWS
EXAMPLE
FUNCTIONS
FAST
IPODS
MANAGE
price
COMING
material
PATTIES
FIGURE
NUMBER
LONG
MONEY
ACCURATELY
previous
important
equations
ITEMS
doing

Figure 5. A word cloud for “why are rates of change useful?”

Stage 4

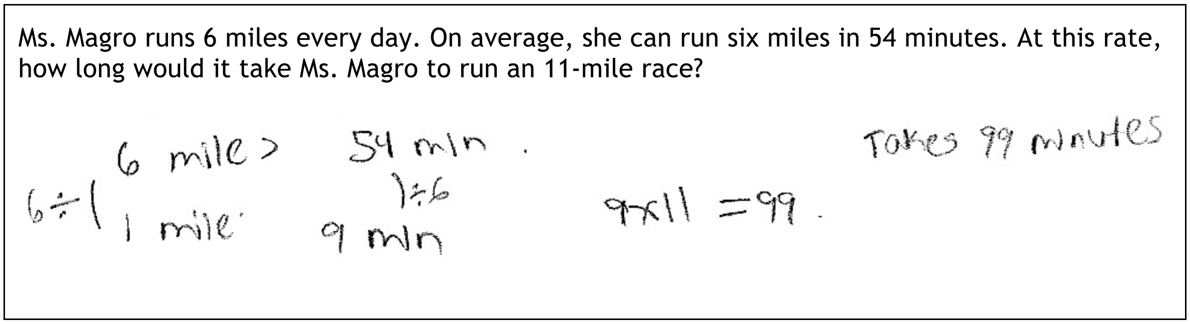
In Stage 4 students reinvented and objectified the unit-rate strategy and the algebraic ratio ($\frac{y_2-y_1}{x_2-x_1}$, the third sub-construct of slope to be reinvented and objectified). I will describe each below.

Reinventing and objectifying the unit rate strategy

The unit rate strategy is a strategy for solving proportional reasoning problems, which involves taking a ratio that represents a many-to-many relationship, scaling it down to a unit rate (a many-per-one relationship) using division, and then scaling the unit rate up to make a new many-per-many relationship using multiplication (Cramer & Post, 1993). Notice that this is very similar to the process of *finding* slope using the algebraic ratio (using division to scale a many-to-many relationship down to many-per-one) and then *using* slope in the equation $y = ax + b$ where the ax scales-up the many-per-one relationship represented by a via multiplication to a new many-per-many relationship. The distinction, of course, is the existence of the constant, b, in the generalized linear function $y = ax + b$. I discuss this later. For now, let's return to the unit rate strategy.

Students reinvented and objectified this strategy while making predictions in proportional relationships (i.e., missing value proportional reasoning problems). The key in designing these problems is that the within-unit pairs should be relatively prime: this is what necessitates the unit rate. For example, as shown in Figure 6, the within-unit pair (6 miles, 11 miles) is relatively prime.

To reinvent and objectify the strategy, students assembled and coordinated ratio tables, the "find one" strategy, and rates of change (objectified as many-per-one). A full analysis of discourse and inscriptions is beyond the scope of this paper. Notice, however, how the student work in Figure 6 demonstrates the assembly and coordination discussed above. On the left side, the student coordinates a ratio table with the "find one" strategy. The result is a rate in the form of a many-per-one relationship. In the center of figure 6, the student brings this objectification into coordination with the objectification of rate as a number that can be accumulated via multiplication to make predictions. This is shown by the multiplication in the center, and the interpretation of the result on the right.



Ms. Magro runs 6 miles every day. On average, she can run six miles in 54 minutes. At this rate, how long would it take Ms. Magro to run an 11-mile race?

6 mile > 54 min .
6÷(1 mile)÷6 9 min
9×11 = 99 .
Takes 99 minutes

Figure 6. Students assembled and coordinated ratio tables, the "find one" strategy and rates of change to reinvent the unit rate strategy.

Reinventing and objectifying the algebraic ratio

The algebraic ratio is a general formula for finding the rate of change, often expressed as $\frac{y_2-y_1}{x_2-x_1}$. In this general formula, one finds differences in the dependent and independent variables, and then divides to create a rate. Up to this point, students had invented and objectified two less general strategies for finding a rate of change. One strategy involved looking at differences in well-ordered tables when the independent variable increased by 1 (e.g., Figure 5). Division was not necessary here because the independent variable increased by one. A second strategy (part of the unit-rate strategy) involved using division to find a rate for situations where the variables were proportional. Subtraction was not necessary here, as there is an implied (0, 0) value (see Figure 6).

For students to reinvent the more general strategy that is captured by the algebraic ratio, they needed to solve problems “just beyond [their] current abilities” where their current strategies would not work, but where a new artifact (the algebraic ratio) would. These problems have five key features, which are demonstrated in the example problem in Figure 7: (1) the students are asked to make a prediction, in (2) a situation where two variables have a linear relationship but (3) are not proportional. Students (4) are given only two points, such that (5) the difference in the independent variable is greater than one. While solving these problems students assembled and coordinated—in various ways—fraction as quotient, the find one strategy, the unit rate strategy, rate of change, and function tables, to invent and objectify the algebraic ratio.

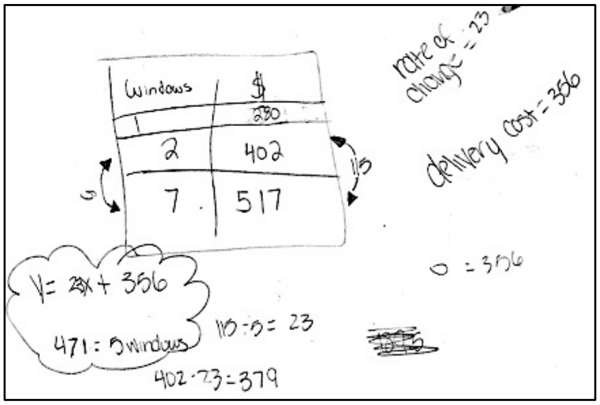
Figure 7 shows two examples of how students assembled and coordinated artifacts while solving a problem to reinvent the algebraic ratio. Group I assembled and brought into coordination function tables and the find one strategy. Group II coordinated fraction as quotient, the unit rate strategy, and a ratio table into a single material assembly. By bringing these two assemblies into coordination, the group realized a contradiction. This motivated them to invent a new strategy, which involved finding the difference in independent and dependent variables, and then using the "find one" strategy on those differences (i.e., the algebraic ratio).

Problem:

Leslie is a window installer. On Friday, she installed two windows, and charged 402 dollars. Last week, on another job, she charged 517 dollars to install seven windows.

A new customer has asked Leslie to install five windows. How much will this cost?

Group I


Windows
$
1
2
402
7
517
5
115
rate of change = 23
delivery cost = 356
0 = 356
y = 23x + 356
471 = 5 windows
115 ÷ 5 = 23
402 − 23 = 379

Group II

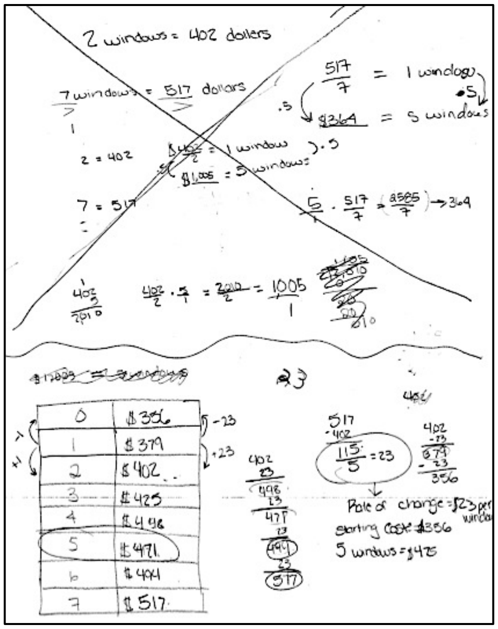

2 windows = 402 dollers
7 windows = 517 dollars
517/7 = 1 window
$364 = 5 windows
2 = 402
7 = 517
0
1
2
3
4
5
6
7
$356
$379
$402
$425
$448
$471
$494
$517
−23
+23
517
−402
115/5 = 23
402
−23
379
−23
356
Rate of change = $23 per window
Starting cost = $356
5 windows = $475

Figure 7. Students assembled and coordinated fraction as quotient, the find one strategy, the unit rate strategy, rate of change, and function tables to reinvent the algebraic ratio.

A detailed analysis of these groups' work and discourse is beyond the limits of this paper. However, it is worth noting the importance of the material instantiation of the artifacts, afforded by the inscriptions each group created. For group I, the inscription of the function table structured the spatial arrangement of the data, which afforded a well-worn way of seeing (notice the similarities in the arrows drawn by Group I in Figure 7, and those drawn by me in Figure 5). For group II, the material inscription enabled the group to bring two assemblies of artifacts into coordination, revealing a contradiction and necessitating the reinvention of the algebraic ratio.

The above discusses how students reinvented the algebraic ratio by assembling and coordinating fraction as quotient, the find one strategy, the unit rate strategy, rate of change, and function tables. In subsequent activity students further objectified the algebraic ratio, including work to distinguish changes from values.

Stage 5

In Stage 5 we introduced the word “slope” and students reinvented and objectified the geometric ratio (“rise over run”; the fourth sub-construct of slope to be reinvented and objectified).

In one sense, the geometric ratio is simply the algebraic ratio in a graph, with $y_2 - y_1$ and $x_2 - x_1$ replaced their geometric equivalents (“rise” and “run” respectively). Thus, we conjectured that students would reinvent the geometric ratio if they solved problems in which they made predictions given a graph. In other words, we conjectured that students would coordinate the algebraic ratio with graphs in the coordinate plane to create the geometric ratio. What we found when students made predictions given a graph was that students did indeed coordinate the algebraic ratio with graphs, just not how we expected.

Rather than reinventing the geometric ratio, students used the graph to extract points from which they either made a table or used the algebraic ratio. For students, the graph was a sign for collection of points. In retrospect, this is not surprising as "graph as a collection of points" is the way that graphs were objectified in stage 2. We realized that, just like students' perception needed to be disciplined so that they could see change in a table, so too did their perception need to be disciplined so that they could see change in a graph. To do this, we introduced a set of tasks that involved showing change on horizontal and vertical number lines with arrows. This activity helped to discipline students' perception to see change in a graph. As shown in Figure 8A, students brought number lines into coordination with coordinate graphs by drawing arrows on the x- and y-axes (shown by the red and blue arrows in Figure 8A). As inscriptions, these arrows could be mobilized to show the traditional "slope triangle," and brought into coordination with other inscriptions to show the connection between slope and rate of change (see Figure 8B).

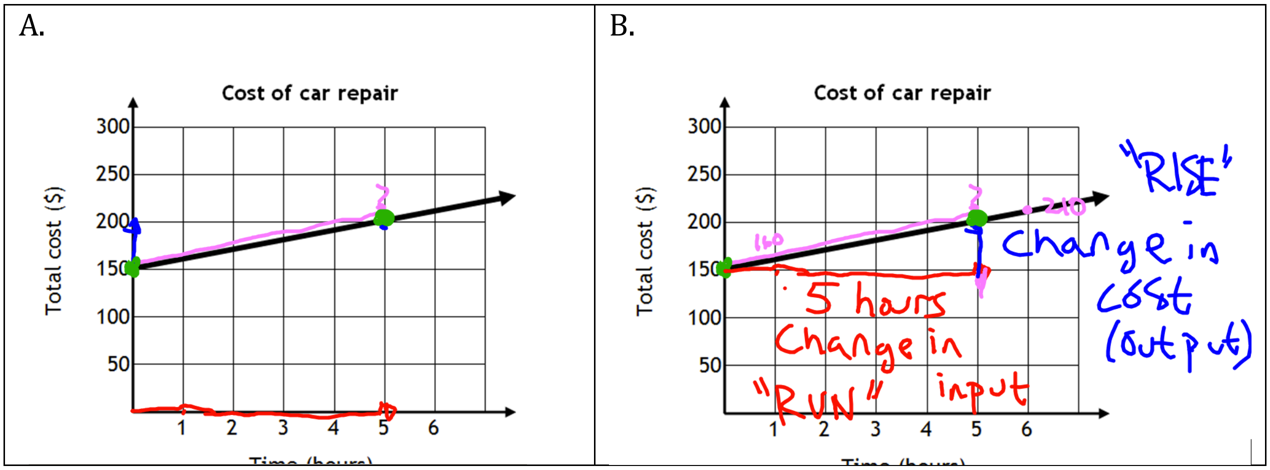


Figure 8. (A) Students brought number lines into coordination with coordinate graphs to show change by drawing arrows on the x- and y-axes. (B) These arrows could be mobilized to form the traditional slope triangle, and brought into coordination with other inscriptions to make the connection between slope and rate of change.

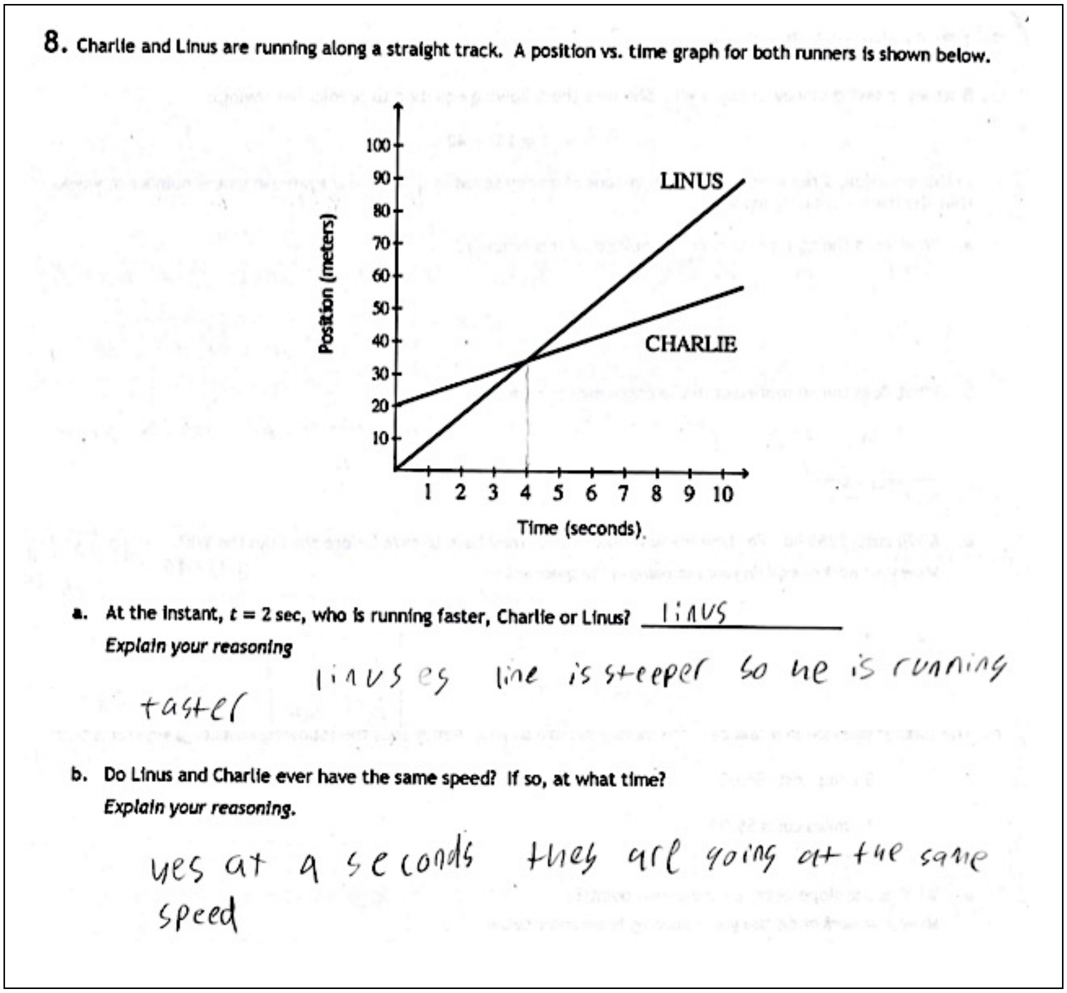
In this way, students objectified coordinate graphs by making horizontal and vertical changes meaningful. In addition, students reinvented the geometric ratio by coordinating number lines, coordinate graphs, and the algebraic ratio. In subsequent activity students further objectified the geometric ratio including work to distinguish change (represented by arrows and the traditional slope triangle), from values (represented by points).

Stage 6

Due to time constraints at the school, we did not have time to implement Stage 6 of the design experiment. Thus, I only reiterate our conjectures, and offer a small amount of supporting evidence from the design experiment and from prior research.

In stage 6, we planned that students would reinvent and objectify the physical property (steepness). We conjecture that students can reinvent the physical property by engaging in activities in which they compare rates for multiple functions graphed in the same coordinate plane. Through these activities, students will reinvent the physical property by coordinating rate of change with graphs in a coordinate plane, and students' perception can be disciplined such that they "see" steepness corresponding to rate of change.

There is some evidence from the research literature and from our design experiment that this reinvention will happen, but that it requires disciplining perception. In the research literature, Tierney and Monk (2007) describe how a class began to reinvent the physical property as the students engaged with a graph of two linear functions. They describe how students disciplined each other's perception so that steepness became salient. In our design experiment, we gave students the problem shown in Figure 9 (inspired by a problem in McDermott, Rosenquist, & Van Zee, 1987) on an individual assessment at the end of the unit. The student response shown in the figure is representative. Many students appealed to steepness to explain that Linus was running faster than Charlie at $t = 2$, however many also suggested that Linus and Charlie were running at the same speed at $t = 4$. Taken together, the prior work and the results from the design experiment suggest that students will coordinate rate of change with coordinate graphs to invent the physical property, but that they should do so in a collaborative environment where they can help to discipline each other's perception.



8. **Charlie and Linus are running along a straight track. A position vs. time graph for both runners is shown below.**

LINUS
CHARLIE
Position (meters)
100
90
80
70
60
50
40
30
20
10
1 2 3 4 5 6 7 8 9 10
Time (seconds)

a. **At the instant, $t = 2$ sec, who is running faster, Charlie or Linus?** linus

Explain your reasoning

linus es line is steeper so he is running faster

b. **Do Linus and Charlie ever have the same speed? If so, at what time?**

Explain your reasoning.

yes at 4 seconds they are going at the same speed

Figure 9. Some evidence that students can reinvent the physical property by coordinating rate of change with graphs in the coordinate plane.

Having reinvented the physical property by coordinating rate of change with coordinate graphs, students can further objectify the physical property by engaging with tasks that involve measuring and comparing the steepness of various objects (similar to those described in Lobato & Thanheiser, 2002).

Conclusion

In this paper I presented a local instructional theory that describes how students reinvented and objectified the sub-constructs of slope as they engaged in activities organized around making predictions. I conceptualized these sub-constructs as mathematical artifacts, and I introduced the notion of a *cascade of artifacts* to explain how students reinvented and objectified new artifacts by bringing other artifacts into coordination. Further, I explained that a considerable amount of objectification involved disciplining perception, that is, learning to “see” particular features in material instantiations of artifacts.

The work was guided by theory and forged in practice. As such, it ought to contribute to both. With respect to practice, teachers can use the LIT as a framework to design instructional sequences that are tailored to their own unique circumstances, and which leverage their students’ unique repertoires. With respect to theory, this work makes two contributions. First, it contributes to our understanding of how students learn about slope, which is a fundamental concept in secondary mathematics. Second, this work builds on and contributes to culturally oriented theories of learning by introducing the notion of a *cascade of artifacts*. This construct has the potential to describe learning in a variety of settings. This should be explored further in future work.

Acknowledgements

I am grateful for the invaluable contributions of Michael Matassa and David Webb, the other members of the research team. I also thank Kris Gutiérrez, Kevin O'Connor, and Susan Jurow, who disciplined my perception to see culture.

References

Brenner, M. E., Mayer, R. E., Moseley, B., Brar, T., Durán, R. P., Reed, B. S., & Webb, D. C. (1997). Learning by understanding: The role of multiple representations in learning algebra. *American Educational Research Journal*, *34*(4), 663–689. doi:10.2307/1163353

Cole, M. (1996). *Cultural psychology: A once and future discipline*. Cambridge, MA: Harvard University Press.

Cole, M. (2010). What's Culture Got to Do With It?: Educational Research as a Necessarily Interdisciplinary Enterprise. *Educational Researcher*, *39*(6), 461–470. doi:10.3102/0013189X10380247

Cole, M., & Wertsch, J. V. (1996). Beyond the individual-social antimony in discussions of Piaget and Vygotsky. *Human Development*, *39*, 250–256.

Confrey, J., Maloney, A., Nguyen, K. H., Mojica, G., & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. In M. Tzekaki, M. Kaldrimidou, & C. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1). Thessaloniki, Greece: PME.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics*, *26*(2), 135–164.

Cramer, K., Bezuk, N., & Behr, M. J. (1989). Proportional relationships and unit rates. *Mathematics Teacher*, *82*(7), 537–544.

Cramer, K., & Post, T. R. (1993). Connecting research to teaching proportional reasoning. *The Mathematics Teacher*, *86*(5), 404–407.

Empson, S. B. (1999). Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. *Cognition and Instruction*, *17*(3), 283–342.

Engeström, Y. (1987). *Learning by expanding: An activity-theoretical approach to developmental research*. Helsinki: Orienta-Konsultit.

Engeström, Y. (2007). Putting Vygotsky to work: The change laboratory as an application of double stimulation. In H. Daniels, M. Cole, & J. V. Wertsch (Eds.), *The Cambridge Companion to Vytgotsky* (pp. 363–382). Cambridge, UK: Cambridge University Press.

Fosnot, C. T., & Dolk, M. (2001a). *Young Mathematicians at Work: Constructing Early Number Sense, Addition, and Subtraction*. Portsmouth, NH: Heinemenn.

Fosnot, C. T., & Dolk, M. (2001b). *Young Mathematicians at Work: Constructing Multiplication and Division*. Portsmouth, NH: Heinemenn.

Fosnot, C. T., & Dolk, M. (2002). *Young Mathematicians at Work: Constructing Fractions, Decimals, and Percents*. Portsmouth, NH: Heinemenn.

Fosnot, C. T., & Jacob, B. (2010). *Young Mathematicians at Work: Constructing Algebra*. Portsmouth, NH: Heinemenn.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics*, *3*(3/4), 413–435.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht, The Netherlands: D. Reidel.

Freudenthal, H. (1987). Mathematics starting and staying in reality. In *Proceedings of the USCMP Conference on Mathematics Education on Development in School Mathematics around the World* (pp. 279–295). Reston, VA: NCTM.

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Gee, J. P. (2011). *How to do Discourse Analysis: A Toolkit*. Taylor & Francis.

Gierl, M. J., Wang, C., & Zhou, J. (2008). Using the attribute hierarchy method to make diagnostic inferences about examinees' cognitive skills in algebra on the SAT ©. *The Journal of Technology, Learning, and Assessment*, *6*(6).

Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education*, *25*(5), 443–471. doi:10.2307/749485

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155–177.

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning*, *6*(2), 105–128. doi:10.1207/s15327833mtl0602

Gravemeijer, K., & Cobb, P. (2006). Design research from a learning design perspective. In J. van den Akker, K. Gravemeijer, S. McKenney, & N. Nieveen (Eds.), *Educational Design Research* (pp. 17–51). New York: Routledge.

Gravemeijer, K., & Terwel, J. (2000). Hans Freudenthal: A mathematician on didactics and curriculum theory. *Journal of Curriculum Studies*, *32*(6), 777–796. doi:10.1080/00220270050167170

Herbert, S. (2008). Where is the rate in the rule? *Australian Senior Mathematics Journal*, *22*(2), 28–37.

Herbert, S., & Pierce, R. (2005). Potential of Technology and a Familiar Context to Enhance Students' Concept of Rate of Change. In *Mathematics Education Research Group of Australasia. Conference (28th: 2005: Melbourne, Vic.)* (pp. 435–442).

Heritage, J., & Clayman, S. (2010). *Talk in Action: Interactions, Identities, and Institutions*. Chichester, UK: Wiley-Blackwell.

Hutchins, E. (1995). *Cognition in the Wild*. Cambridge, MA: MIT Press.

Hutchins, E., & Palen, L. (1997). Constructing meaning from space, gesture, and speech. In L. B. Resnick, R. Säljö, C. Pontecorvo, & B. Burge (Eds.), *Discourse, Tools, and Reasoning: Essays on Situated Cognition* (pp. 23–40). London: Springer.

Jaworski, A., & Coupand, N. (2006). Perspecitves on discourse analysis. In A. Jaworski & N. Coupand (Eds.), *The Discourse Reader* (2nd ed., pp. 1–37). New York: Routledge.

Karplus, R., Pulos, S., & Stage, E. K. (1983). Early adolescents' proportional reasoning on "rate" problems. *Educational Studies in Mathematics*, *14*(3), 219–233.

Kieren, T. (1980). The rational number construct: Its elements and mechanisms. In T. Kieren (Ed.), *Recent Research on Number Learning* (pp. 125–149). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Lane, S. (2008, August). Apple already building iPhones at rate of 40 million a year? *Apple Insider*.

Latour, B. (1990). Visualisation and cognition: Drawing things together. In M. Lynch & S. Woolgar (Eds.), *Knowledge and Society: Studies in the Sociology of Culture and Present* (Vol. 6). Cambridge, MA: MIT Press.

Leont'ev, A. N. (1981). An outline of the evolution of the psyche. *Problems of the Development of the Mind, Progress, Moscow*, 156–272.

Lobato, J., & Ellis, A. B. (2002). The teacher's role in supporting students' connections between realistic situations and conventional symbol systems. *Mathematics Education Research Journal*, *14*(2), 99–120. doi:10.1007/BF03217356

Lobato, J., Ellis, A. B., & Munoz, R. (2003). How "focusing phenomena" in the instructional environment support individual students' generalizations. *Mathematical Thinking and Learning*, *5*(1), 1–36.

Lobato, J., & Thanheiser, E. (2002). Developing understading of ratio-as-measure as a foundation for slope. In B. Litwiller & G. Bright (Eds.), *Making Sense of Fractions, Ratios,* *and Proportions* (pp. 162–175). Reston, VA: National Council of Teachers of Mathematics.

McDermott, L. C., Rosenquist, M. L., & Van Zee, E. H. (1987). Student difficulties in connecting graphs and physics: Exampies from kinematics. *American Journal of Physics*, *55*(6), 503–513.

Middleton, J. A., & van den Heuvel-Panhuizen, M. (1995). The ratio table. *Mathematics Teaching in the Middle School*, *1*(4), 282–288.

NCTM. (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Nemirovsky, R. (1996). Mathematical narratives, modeling, and algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to Algebra: Perspectives for Research and Teaching* (pp. 197–220). Dordrecht, The Netherlands: Kluwer Academic Publishers.

NGA. (2010). *Common Core Standards for Mathematics*. Washington DC.

Nunes, T., Desli, D., & Bell, D. (2003). The development of children's understanding of intensive quantities. *International Journal of Educational Research*, *39*(7), 651–675. doi:10.1016/j.ijer.2004.10.002

Potter, J. (1996). *Representing Reality: Discourse, Rhetoric, and Social Construction*. London: SAGE Publications.

Radford, L. (2002). The seen, the spoken, and the written: A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics*, *22*(2), 14–23.

Radford, L. (2008a). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp. 215–234). Rotterdam: Sense Publishing.

Radford, L. (2008b). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, *70*(2), 111–126. doi:10.1007/s10649-008-9127-3

Radford, L., Bardini, C., & Sabena, C. (2007). Perceiving the general: The multisemiotic dimension of students' algebraic activity. *Journal for Research in Mathematics Education*, *38*(5), 507–530.

Schliemann, A. D., & Carraher, D. W. (2000). When tables become function tables. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*. Utrecht, The Netherlands: The Freudenthal Institue.

Simon, M. A., & Blume, G. W. (1994). Mathematical modeling as a component of understanding ratio-as-measure: A Study of prospective elementary teachers. *The Journal of Mathematical Behavior*, *13*(2), 183–197.

Stevens, R., & Hall, R. (1998). Disciplined perception: Learning to see in technoscience. In M. Lampert & M. L. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 107–149). Cambridge, UK: Cambridge University Press.

Streeck, J. (2009). *Gesturecraft: The manu-facture of meaning*. Amsterdam: John Benjamins.

Streeck, J., Goodwin, C., & LeBaron, C. (2011). Embodied interaction in the material world: An introduction. In J. Streeck, C. Goodwin, & C. LeBaron (Eds.), *Embodied Interaction: Language and Body in the Material World* (pp. 1–26). Cambridge, UK: Cambridge University Press.

Streefland, L. (1993). Fractions: A realistic approach. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational Numbers: An Integration of Research* (pp. 289–326). Hillsdale, NJ: Erlbaum.

Stroup, W. M. (2002). Understanding qualitative calculus: A structural synthesis of learning research. *International Journal of Computers for Mathematical Learning*, *7*, 167–215.

Stump, S. L. (1999). Secondary Mathematics Teachers' Knowledge of Slope. *Mathematics Education Research Journal*, *11*(2), 124–144.

Stump, S. L. (2001). High school precalculus students' understanding of slope as measure. *School Science and Mathematics*, *101*(2), 81–89. doi:10.1111/j.1949-8594.2001.tb18009.x

Thompson, P. W. (1994a). Images of Rate and Operational Understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics*, *26*(2), 229–274.

Thompson, P. W. (1994b). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 179–234). Albany, NY: SUNY Press.

Tierney, C., & Monk, S. (2007). Children's reasoning about change over time. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the Early Grades* (pp. 185–200). Lawrence Erlbaum.

Treffers, A. (1987). *Three Dimensions: A Model of Goal and Theory Description in Mathematics Instruction. The Wiskobas Project*. Dordrecht, The Netherlands: Reidel.

Vygotsky, L. S. (1978). *Mind in society*. (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds.). Cambridge, MA: Harvard University Press.

Wartofsky, M. W. (1979). Perception, representation, and the forms of action: Towards an historical epistemology. In M. W. Wartofsky (Ed.), *Models: Representation and the scientific understanding*. Dordrecht, The Netherlands: D. Reidel.

Wertsch, J. V. (1998). *Mind as action*. Oxford, UK: Oxford University Press.

Wilson, P. H., Edgington, C. P., Nguyen, K. H., Pescosolido, R. C., & Confrey, J. (2011). Fractions: How to share fair. *Mathematics Teaching in the Middle School*, *17*(4), 230–236.

Wittgenstein, L. (1958). *Philosophical investigations* (Vol. 255). Oxford, UK: Blackwell.

Yerushalmy, M. (1997). Mathematizing Verbal Descriptions of Situations: A Language to Support Modeling. *Cognition and Instruction*, *15*(2), 207–264.