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Modeling Your Way to Understanding with Realistic Mathematics Education

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A classic RME task

4. What is the price of one umbrella? One cap?

(*Mathematics in Context*, "Comparing Quantities")

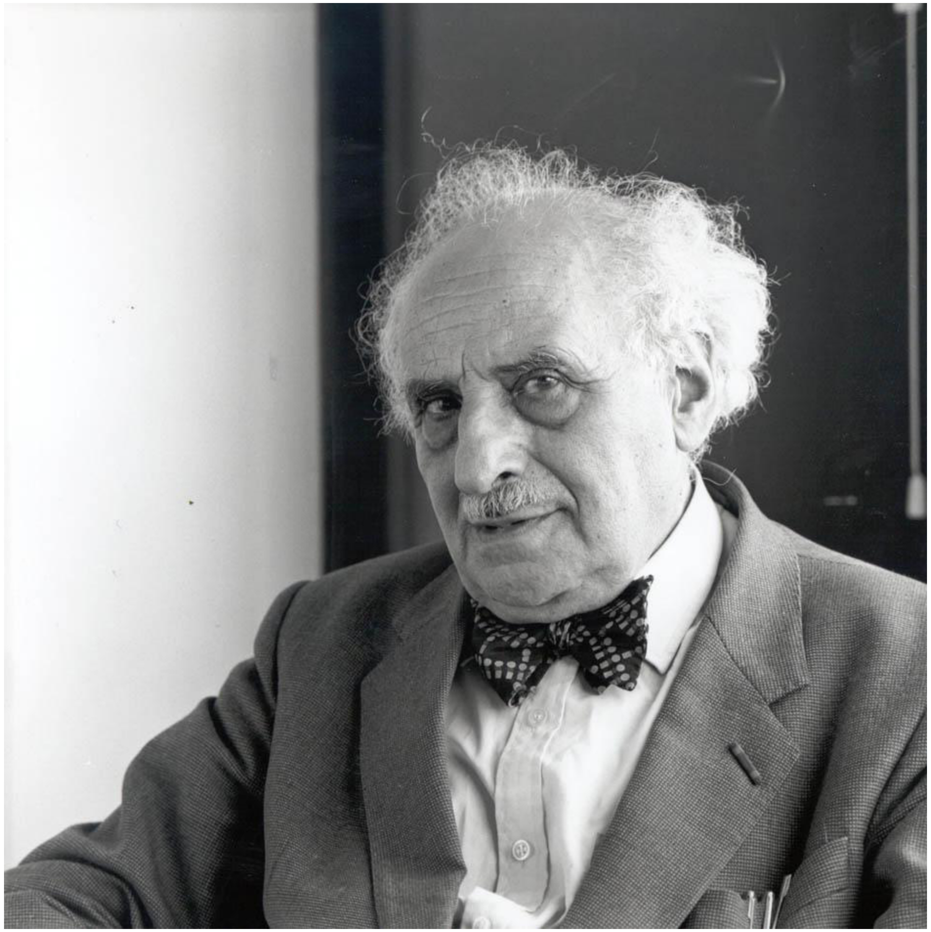
You can download these slides: http://mathed.net/nctm16

Session outline

1. What is Realistic Mathematics Education?
2. What are some perspectives on modeling?
3. A high school example of models and progressive formalization
4. Design time
    a. What models might you use?
    b. How might they be sequenced?

Realistic Mathematics Education

A little history of RME



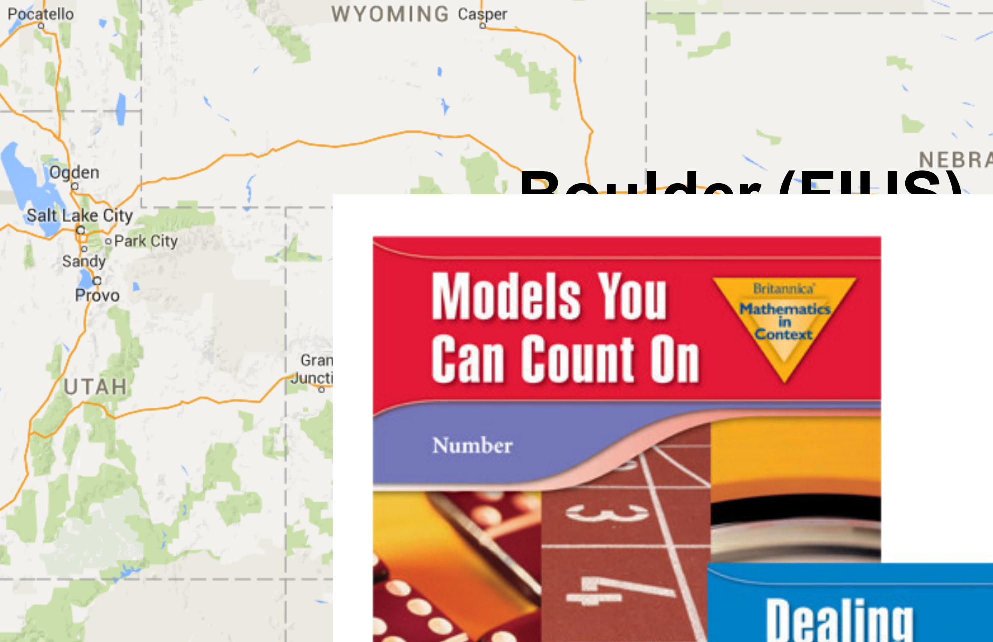
Hans Freudenthal (1905-1990)

The need for a new math, but not *that* New Math.

Why to teach mathematics so as to be useful

"Systematization is a great virtue of mathematics, and if possible, the student has to learn this virtue, too. But then I mean the activity of systematizing, not its result. Its result is a system, a beautiful closed system, closed with no entrance and no exit. In its highest perfection it can even be handled by a machine. But for what can be performed by machines, we need no humans. What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics."

- Hans Freudenthal (1968)



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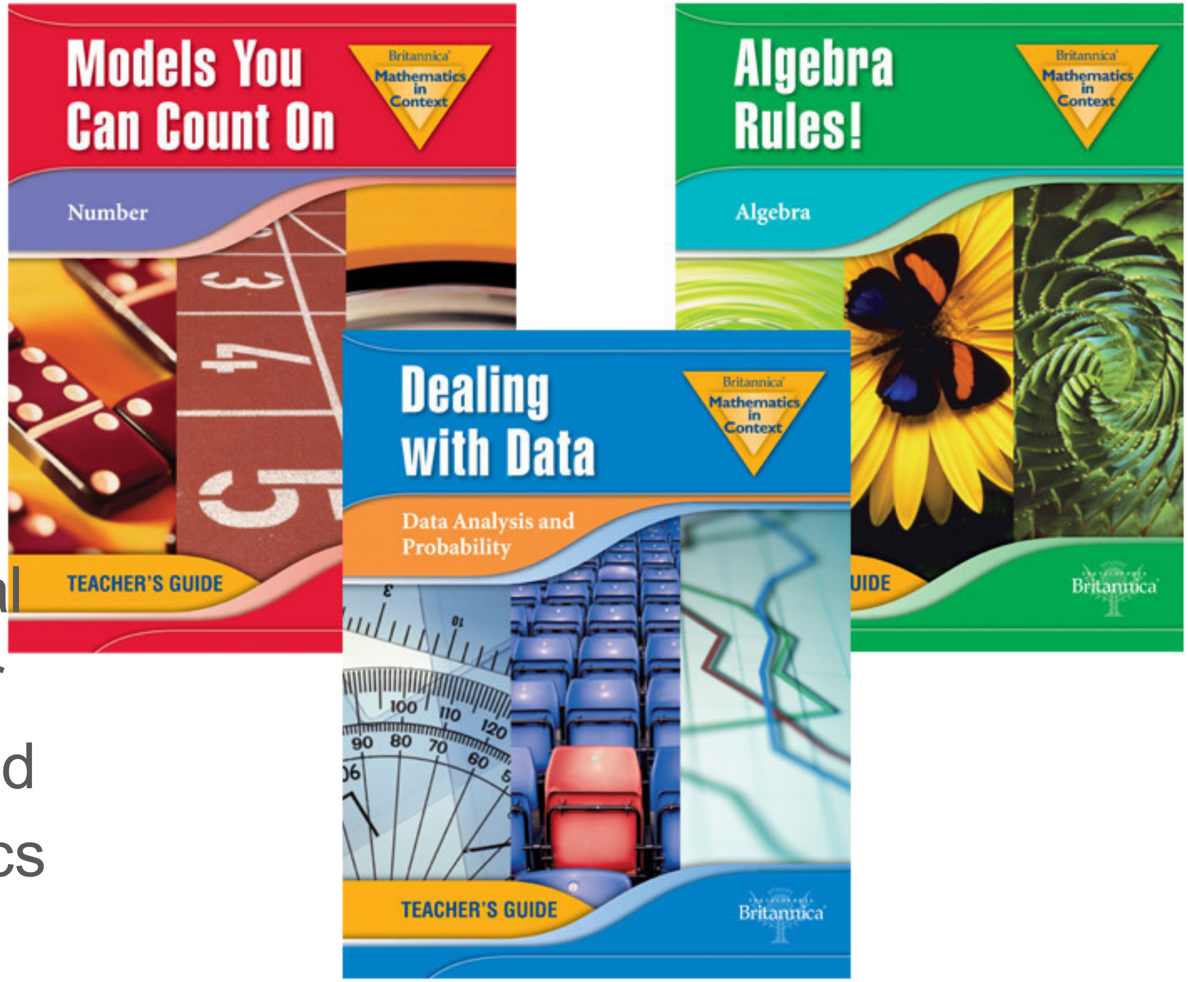
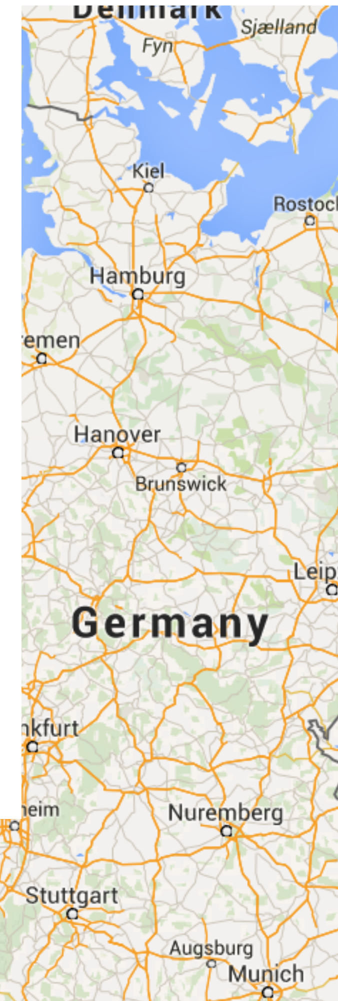
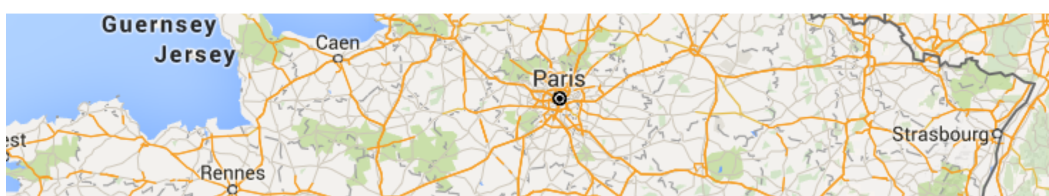






Freudenthal Institute for Science and Mathematics Education

Six Principles of RME

1. Activity principle: Mathematics is a human activity of *mathematizing* the world
2. Reality principle: “Realistic” -> “to imagine”
3. Guided reinvention principle: not “discovery,” not direct instruction
4. Level principle: Move from informal, context-specific to formal mathematics
5. Intertwinement principle: Integration of mathematical concepts
6. Interaction principle: Learning is social with individual

Perspectives on Modeling

These aren’t the models you’re looking for

1. Modeling as demonstrating (I do, you do)
2. Modeling as scaled object
3. Modeling = manipulatives
4. Modeling = “real world”

What the Common Core says about modeling

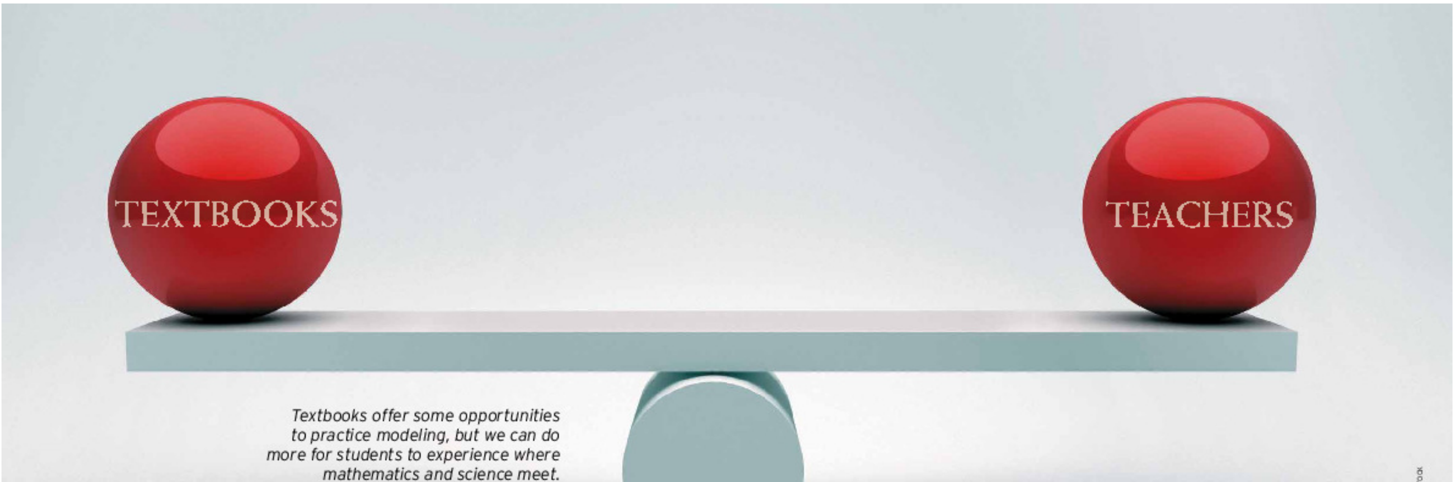
“Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions.” (CCSSI, p. 72)

From the GAIMME Report

"Mathematical modeling is a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena." (p. 10)

http://www.comap.com/Free/GAIMME/

Modeling, Meyer-style



Textbooks offer some opportunities to practice modeling, but we can do more for students to experience where mathematics and science meet.

MISSING THE PROMISE OF Mathematical Modeling

Dan Meyer

The Common Core State Standards for Mathematics (CCSSM) have exerted enormous pressure on every participant in a child's education. Students are struggling to meet new standards for mathematics learning, and parents are struggling to understand how to help them. Teachers are growing in their capacity to develop new mathematical competencies, and administrators are growing in their capacity to support them.

These standards have also exerted pressure on textbook publishers, who must provide curriculum that aligns with the CCSSM. The CCSSM have made some of this existing content obsolete or pushed it to other grade levels. In other cases, publishers have had to develop new content aligned to the CCSSM. But a recent study of fourth-grade textbooks found that this alignment has been slippery, with many textbooks including content external to the CCSSM, failing to include critical CCSSM content or duplicating their previous unaligned editions to an inappropriate degree (Polikoff 2014).

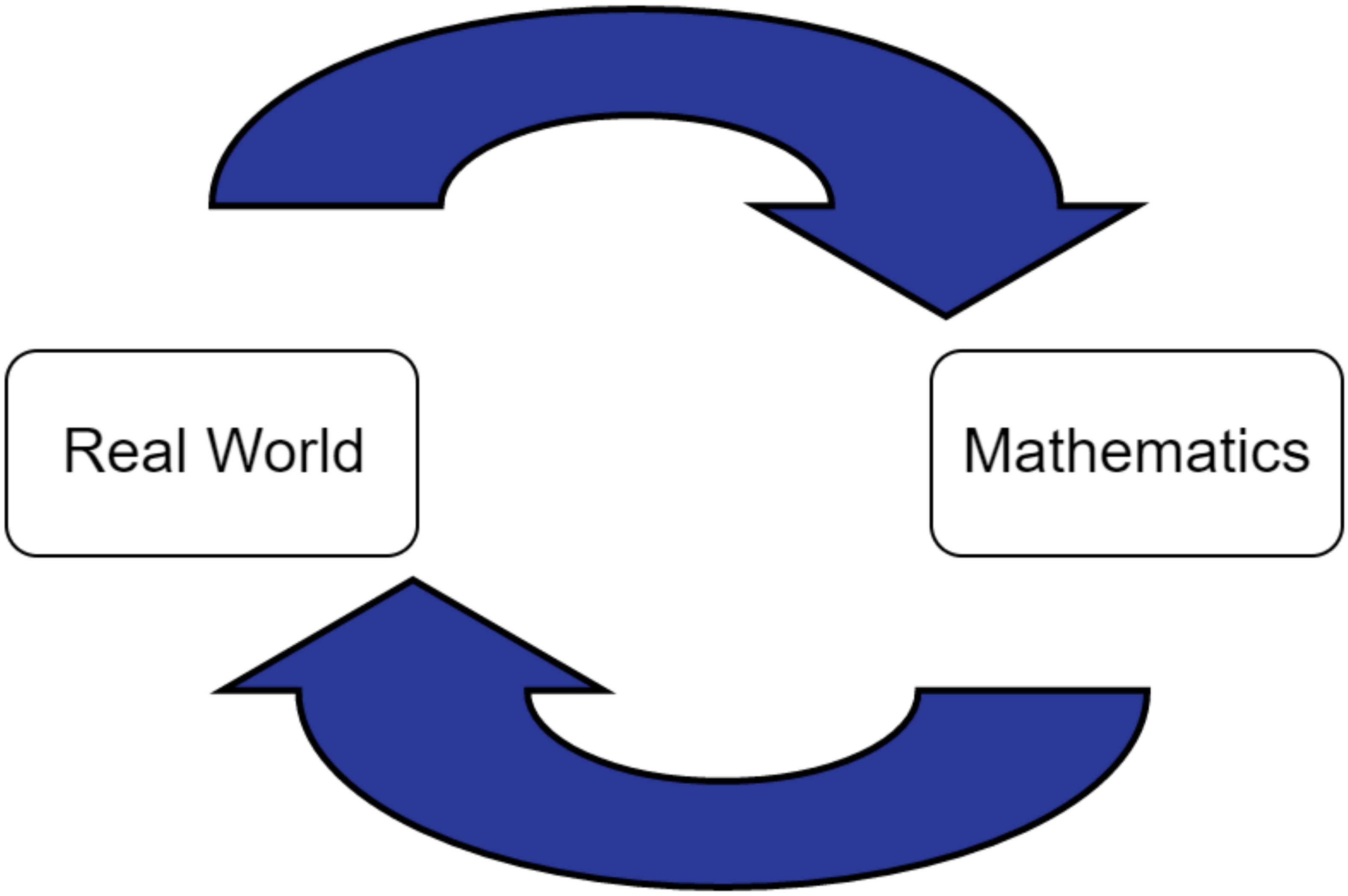
This situation should concern us all given the large sums of money spent nationally on textbooks and the high degree to which teachers take their instructional cues from textbooks. What incentives do publishers have to undertake these costly alignments and developments? The CCSS issued a Publishers' Criteria, but these criteria are not binding in any sense. We—the people who buy textbooks or influence those who do—are publishers' only incentive.

With that rationale in mind, what follows is my analysis of how well textbooks fulfill the promise of one particular standard—mathematical modeling—as it is represented in the CCSSM.

I choose to examine modeling for several reasons. First, speaking strictly personally, I studied mathematics as a child and mathematics education as an adult because of powerful experiences I had using mathematics as a model for the world around me. I want students to have similar experiences. Second, in my work with teachers in professional development, I find modeling with mathematics (Standard for Mathematical Practice 4, CCSSI 2010, p. 7) to be one of the practice standards most in need of explication. Five different teachers may have five different understandings of its meaning. Third, mathematical modeling is the standard where mathematics and science meet. The practice standards of the Next Generation Science Standards (NSTA 2012) resemble the mathematical modeling standards of the CCSSM so closely that we should ensure that we get our end right.

In my research, I analyzed two textbooks in particular—an algebra 1 textbook and a geometry textbook, both published by McGraw-Hill (Carter et al. 2013a, 2013b). I chose these particular texts

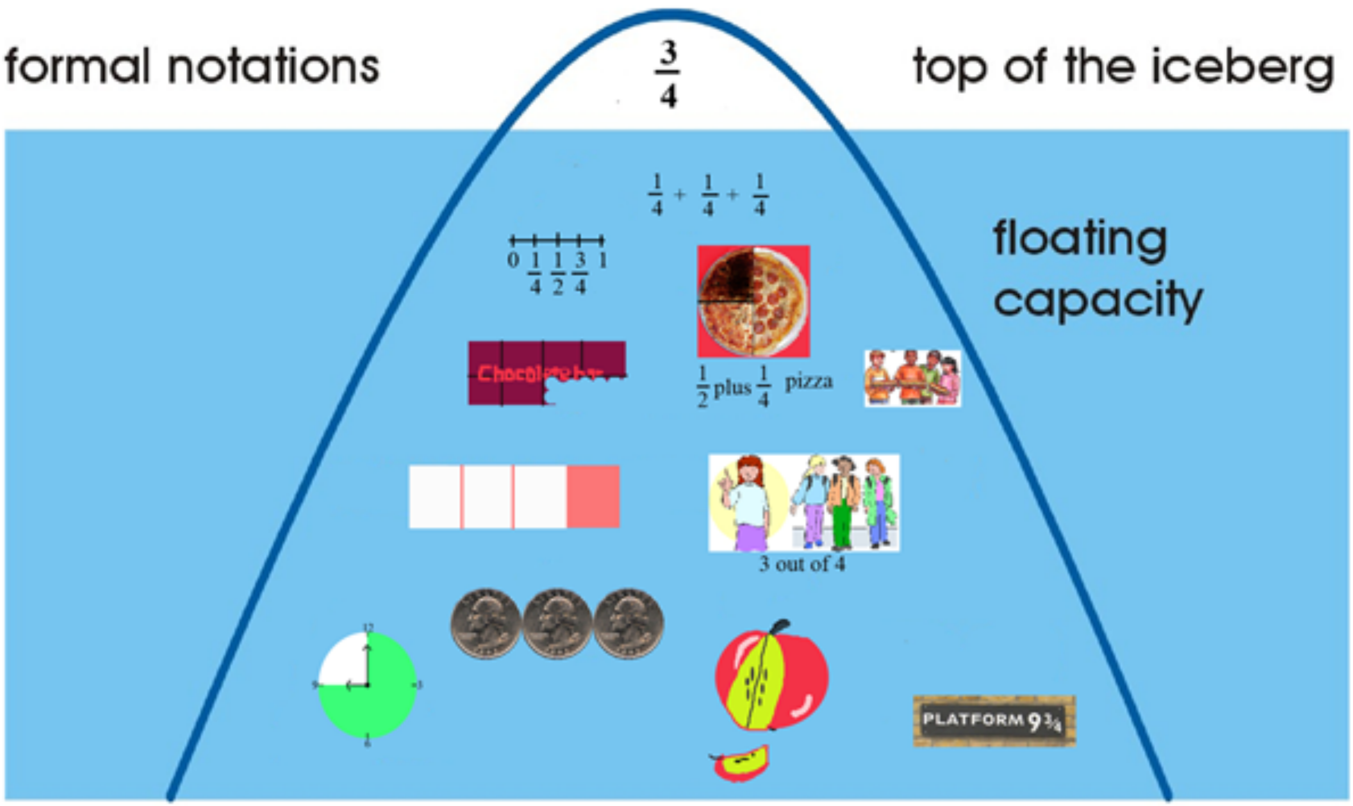
Modeling as a form of translation



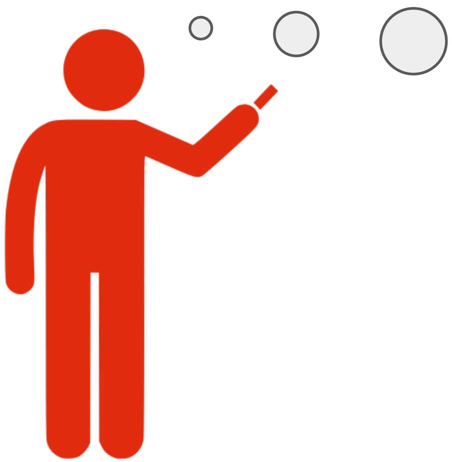
RME: Modeling as a form of organizing

"The idea is that informal ways of modelling emerge when the students are organizing contextual problems. Later, these ways of modelling serve as a basis for developing formal mathematical knowledge. To be more precise, at first a model is constituted as a context-specific model *of* acting in a situation, then the model is generalized over situations. Thus, the model changes character, it becomes an entity of its own, and in this new shape it can function as a model *for* more formal mathematical reasoning." (Gravemeijer, 1997, p. 394)

Progressive formalization & iceberg metaphor



Worried about putting “word problems” first?



Contextual problems are harder than symbolic algebra

Empirically: Contextual problems are *easier* that symbolic algebra for students

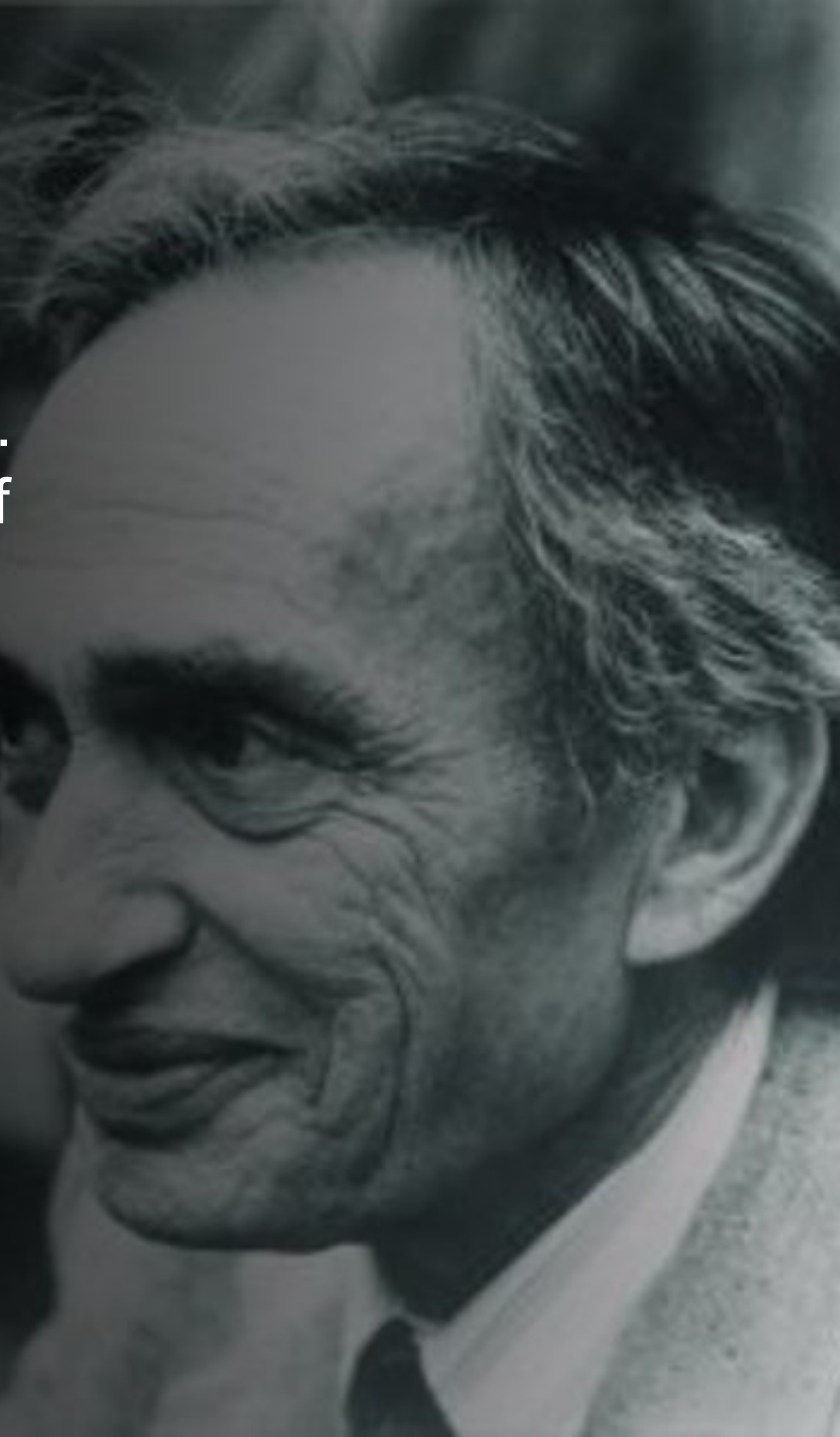
Nathan & Koedinger, 2000

Binomials, Quadratics, & Polynomials

A Traditional Approach

"Very early in our mathematical education – in fact in junior high school or early in high school itself – we are introduced to polynomials. For a seemingly endless amount of time we are drilled, to the point of utter boredom, in factoring them, multiplying them, dividing them, simplifying them. Facility in factoring a quadratic becomes confused with genuine mathematical talent."

– I. N. Herstein, Topics in Algebra (1975, p. 153)



A procedural approach to polynomials

Multiplication: $(x + 5)(x + 3) = x^2 + 8x + 15$ FOIL

Factoring: $x^2 + 6x + 5 = (x + 5)(x + 1)$ guess and check

Complete the square: $x^2 + 4x + 1 = (x + 2)^2 - 3$ algorithm

Polynomial division:

$$\frac{x^2 + 9x + 8}{x + 2} = (x + 7)(x + 2) - 6$$

long/synthetic division

Context: Projectile motion

Modeling as a form of translation? *Context is good*

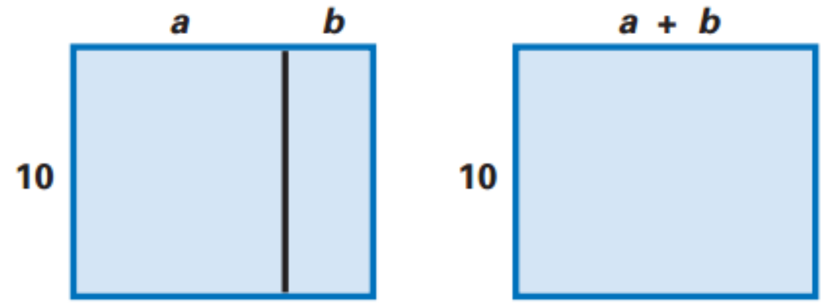
Modeling as a form of organization? *Context is…?*

Binomials, Quadratics, & Polynomials

An RME Approach

Equivalent Expressions

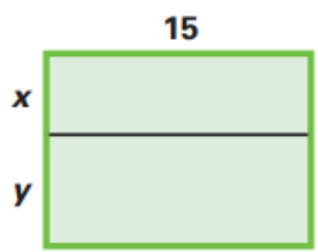
$10a + 10b$ is equivalent to $10(a + b)$.



12. How can you explain this using the picture?

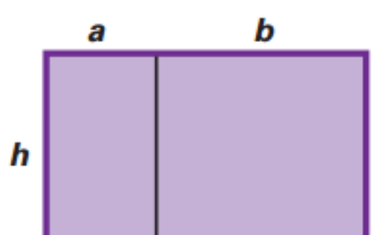
13. Find equivalent expressions using the pictures.

a. $15x + 15y =$ ....

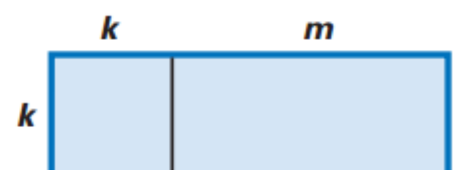


b. ....

c. ....

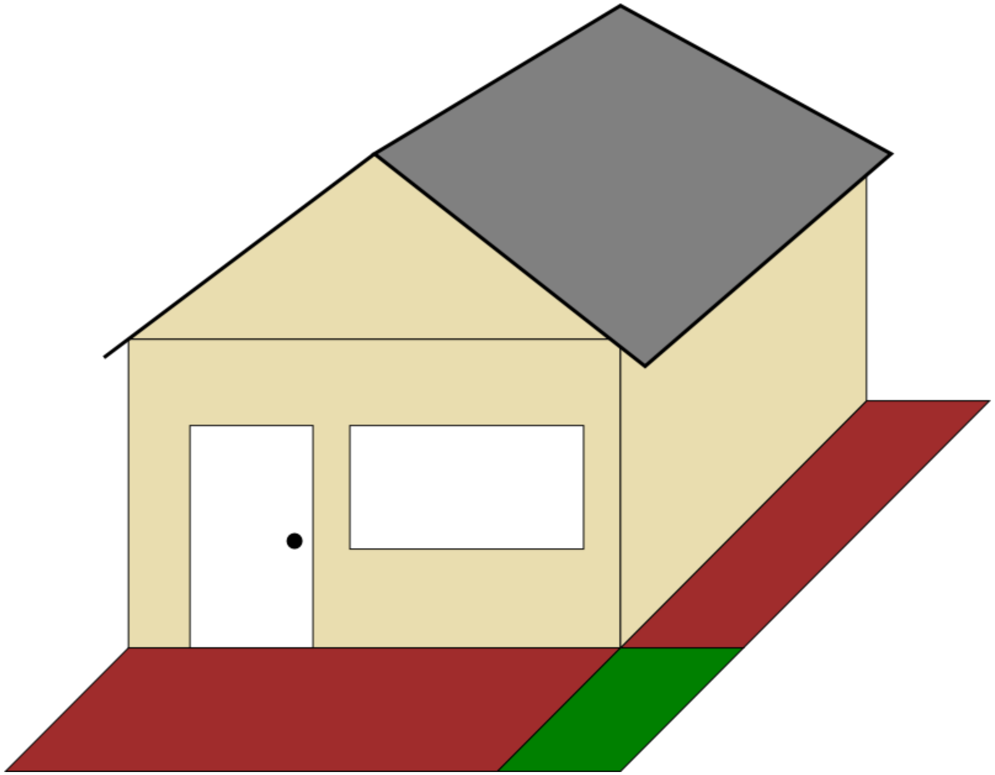


d. ....



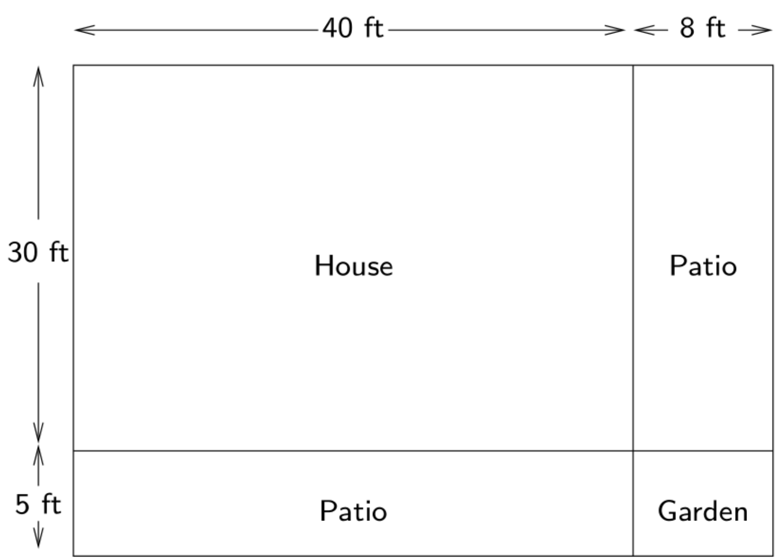
(*Algebra Rules*, Mathematics in Context, 2005, p. 47)

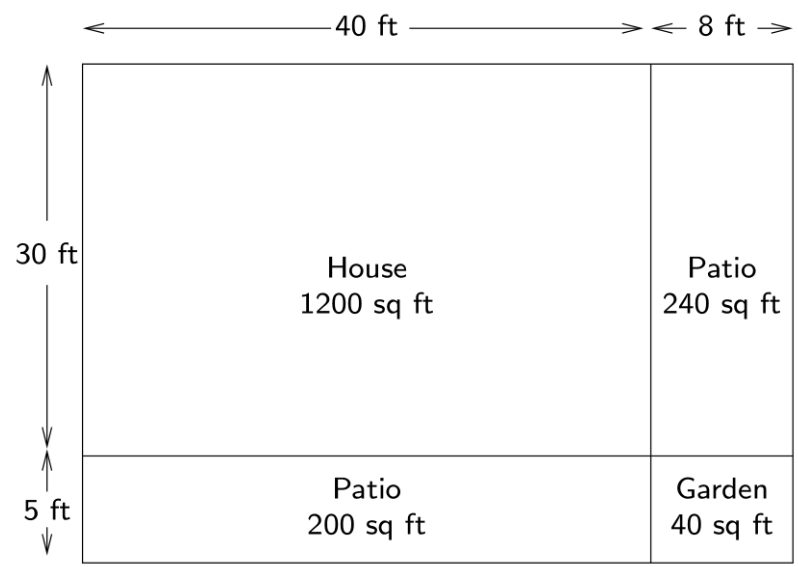
Context



Building on partial products and prior work

Given the dimensions of Peter's house and patios, find the area of the house, each patio, and garden.

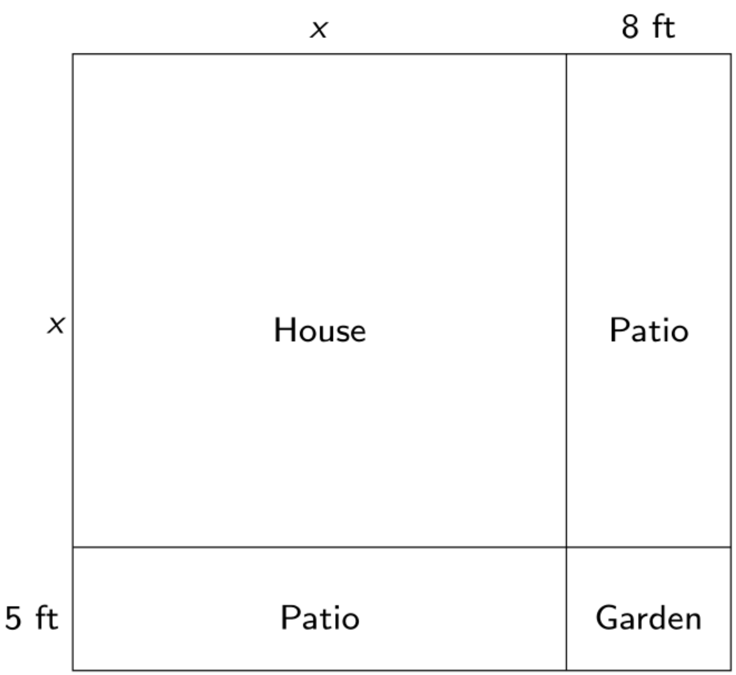




What is the total area of the house, patios, and garden? Can you find it more than one way?

Building on partial products and prior work

Peter's friend Amy wants to have patios and a garden, too. Peter knows Amy's house is square, but doesn't know how big, so he just labels the length and width of Amy's house *x*.



How can you write the area of the house?

$x * x = x^2$ sq ft

How can you write the area of each patio?

8 * x = 8 sq ft and 5 * x = 5x sq ft

What is the area of the garden?

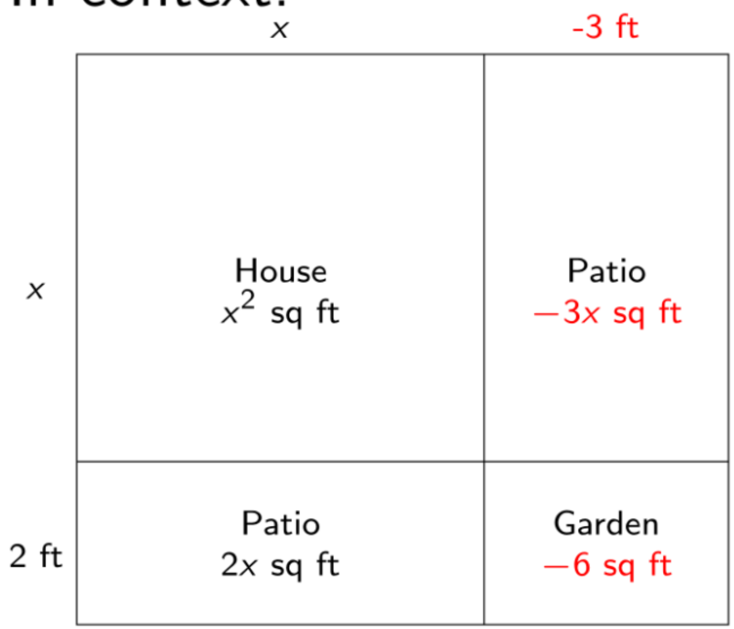
5 * 8 = 40 sq ft

Can you write the total area of the house, patios, and garden more than one way?

Models *of* to models *for*

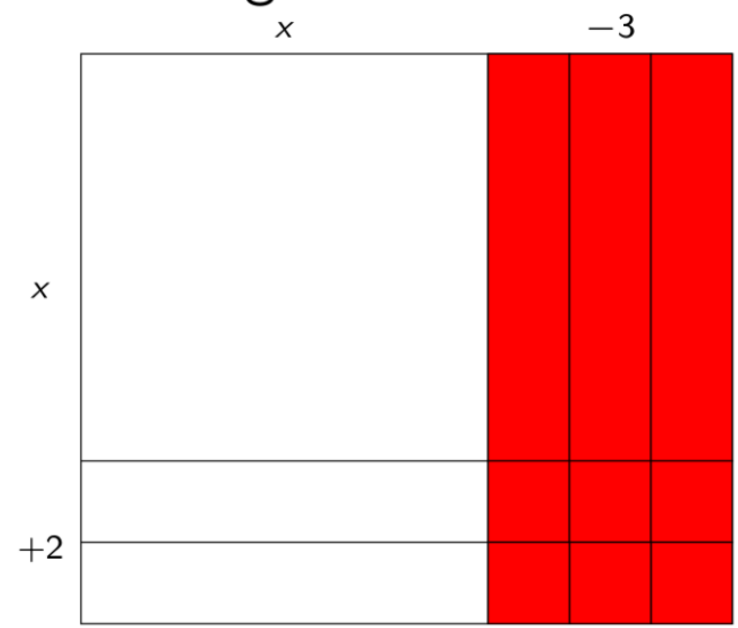
Consider the multiplication of $(x + 2)(x - 3)$.

In context:



(Not "realistic")

With Algebra Tiles:



(Still useful)

Models *of* to models *for*

Consider the multiplication of $(x^2 + 4x + 1)(3x + 4)$.

With Algebra Tiles:

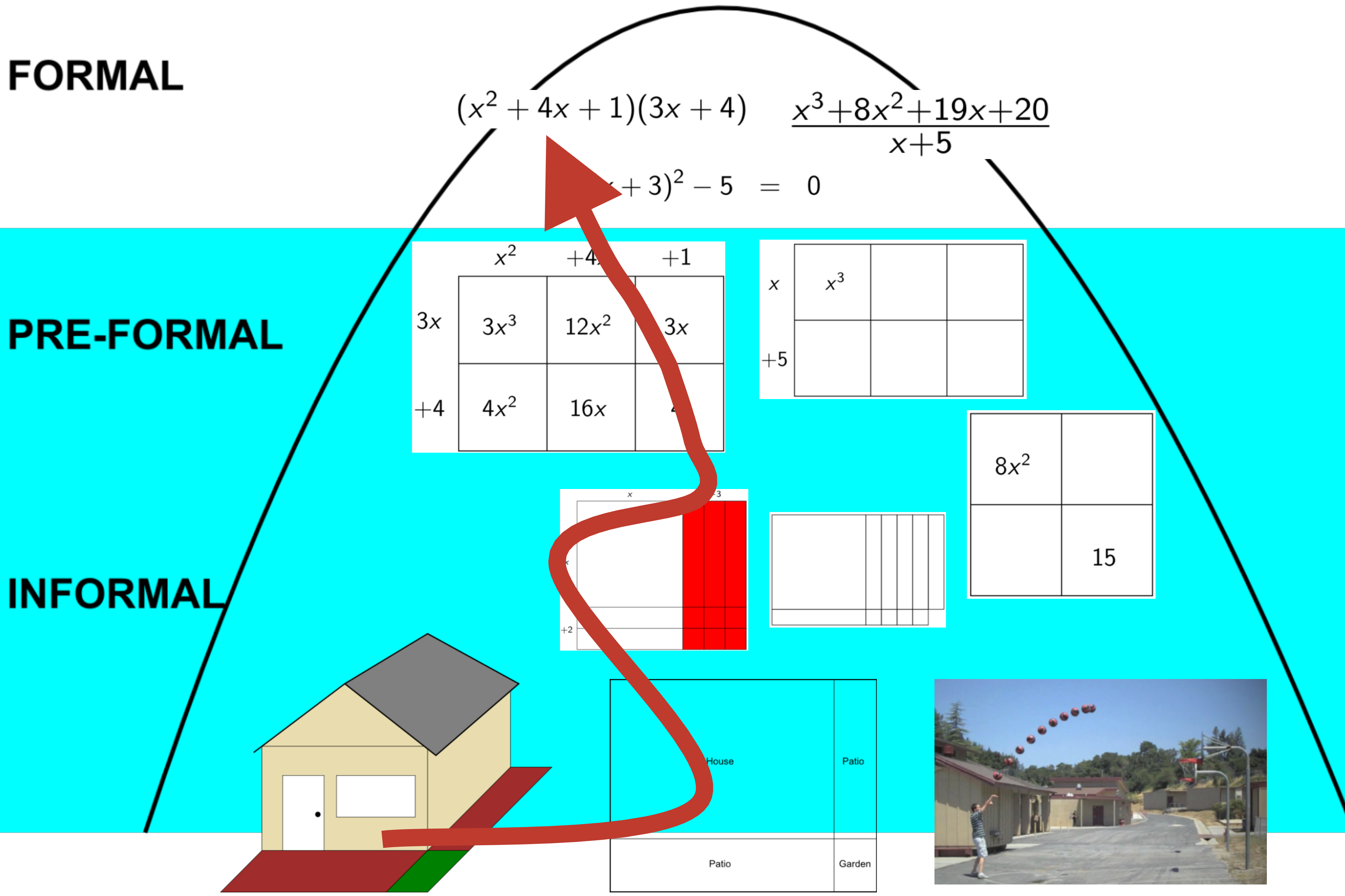
Dimensions?

With Box/Table:

	x^2	$+4x$	$+1$
$3x$	$3x^3$	$12x^2$	$3x$
$+4$	$4x^2$	$16x$	4

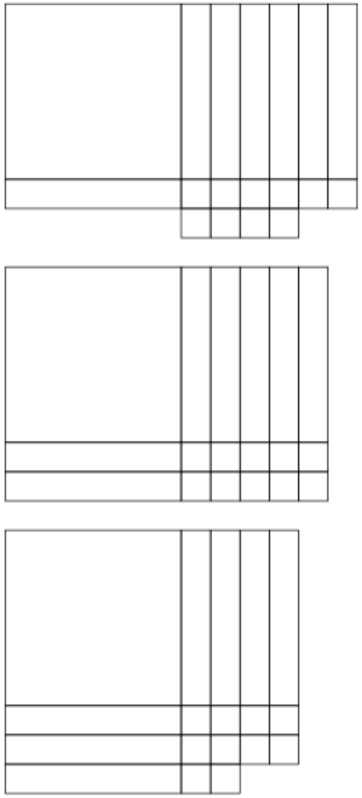
Multiplying problem string

1. $(x+6)(x+7)$
2. $(2x+3)(x+9)$
3. $(-3x+7)(2x-4)$


FORMAL
PRE-FORMAL
INFORMAL
$(x^2 + 4x + 1)(3x + 4)$
$\frac{x^3+8x^2+19x+20}{x+5}$
$+ 3)^2 - 5 = 0$
x^2
$+1$
$3x$
$3x^3$
$12x^2$
$3x$
$+4$
$4x^2$
$16x$
x
x^3
$+5$
$8x^2$
15
x
$+2$
House
Patio
Patio
Garden

What are the dimensions?

Robin wants patios and a garden next to his house, arranged in a rectangle the same way Peter and Lisa have theirs arranged. Robin has x2 + 7x +10 sq ft of space. Use Algebra Tiles to model Robin's house, patios, and gardens.



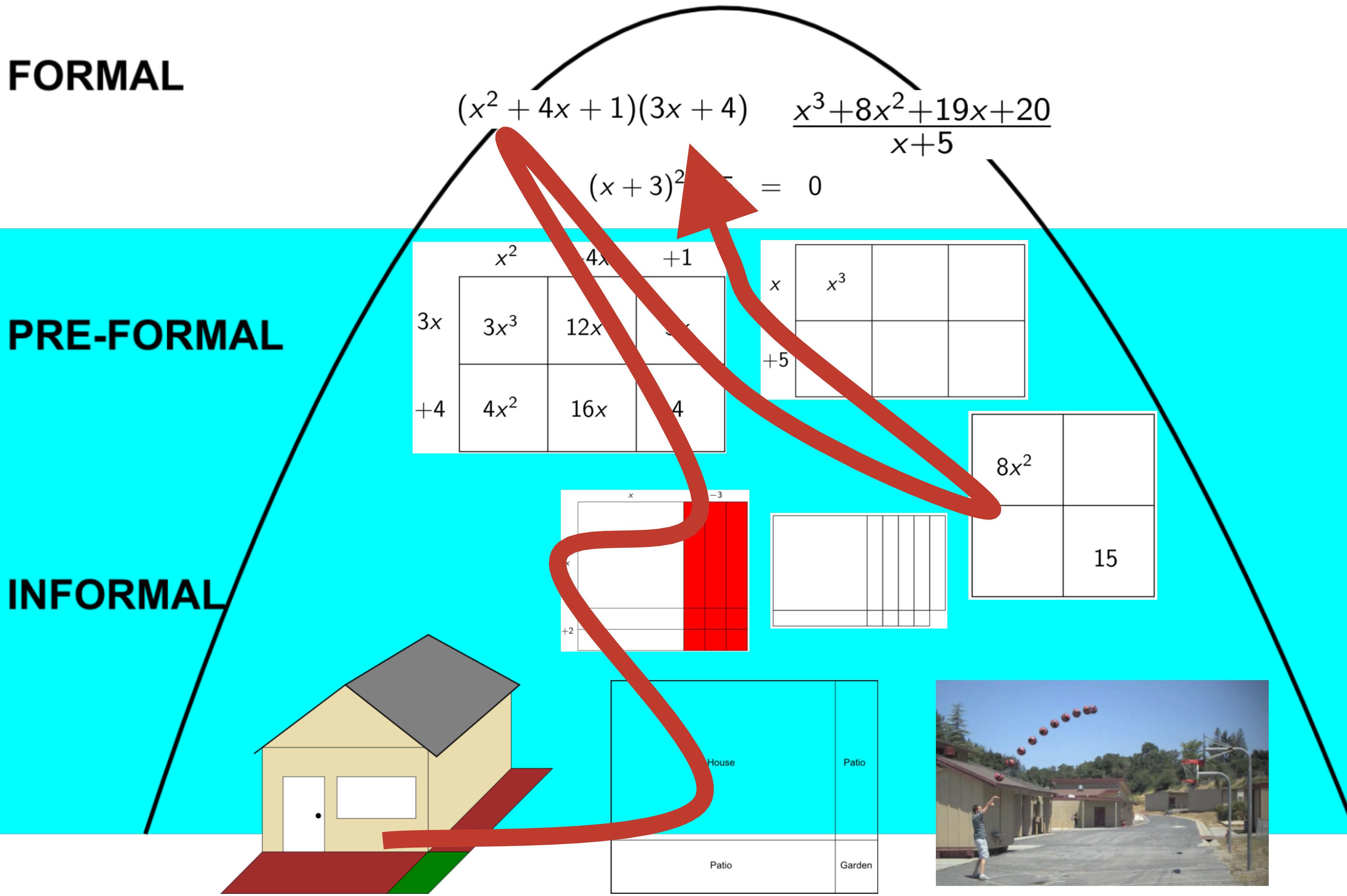
Which arrangement makes a rectangle? What are its dimensions?

Factoring problem string

1. $x^2 + 4x + 3$
2. $x^2 + 6x + 8$
3. $2x^2 + 7x + 3$

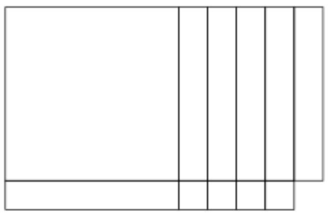
Model for #3

$2x^2$	
	3

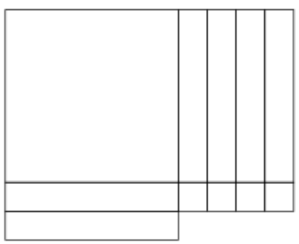

FORMAL
PRE-FORMAL
INFORMAL
$(x^2 + 4x + 1)(3x + 4)$
$\frac{x^3+8x^2+19x+20}{x+5}$
$(x+3)^2$
$= 0$
x^2
$+1$
$3x$
$3x^3$
$12x$
$+4$
$4x^2$
$16x$
4
x
x^3
$+5$
$8x^2$
15
House
Patio
Patio
Garden

What if we can't make a rectangle with tiles?

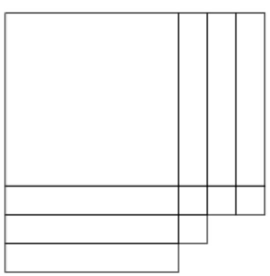
If you're trying to solve $x^2 + 6x + 4 = 0$, which of these equivalencies offers a way forward?



$$x^2 + 6x + 4 \iff (x+5)(x+1) - 1$$



$$x^2 + 6x + 4 \iff (x+4)(x+2) - 4$$



$$x^2 + 6x + 4 \iff (x+3)^2 - 5$$

Completing the square

Completing the square to solve $x^2 + 6x + 4 = 0$:

	x	
x	x^2	?
	?	

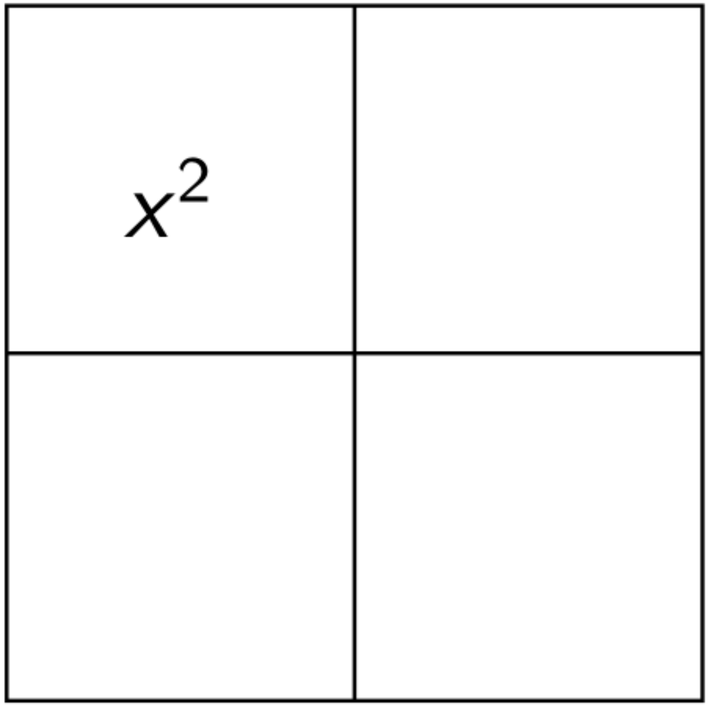
	x	$+3$
x	x^2	$3x$
$+3$	$3x$	

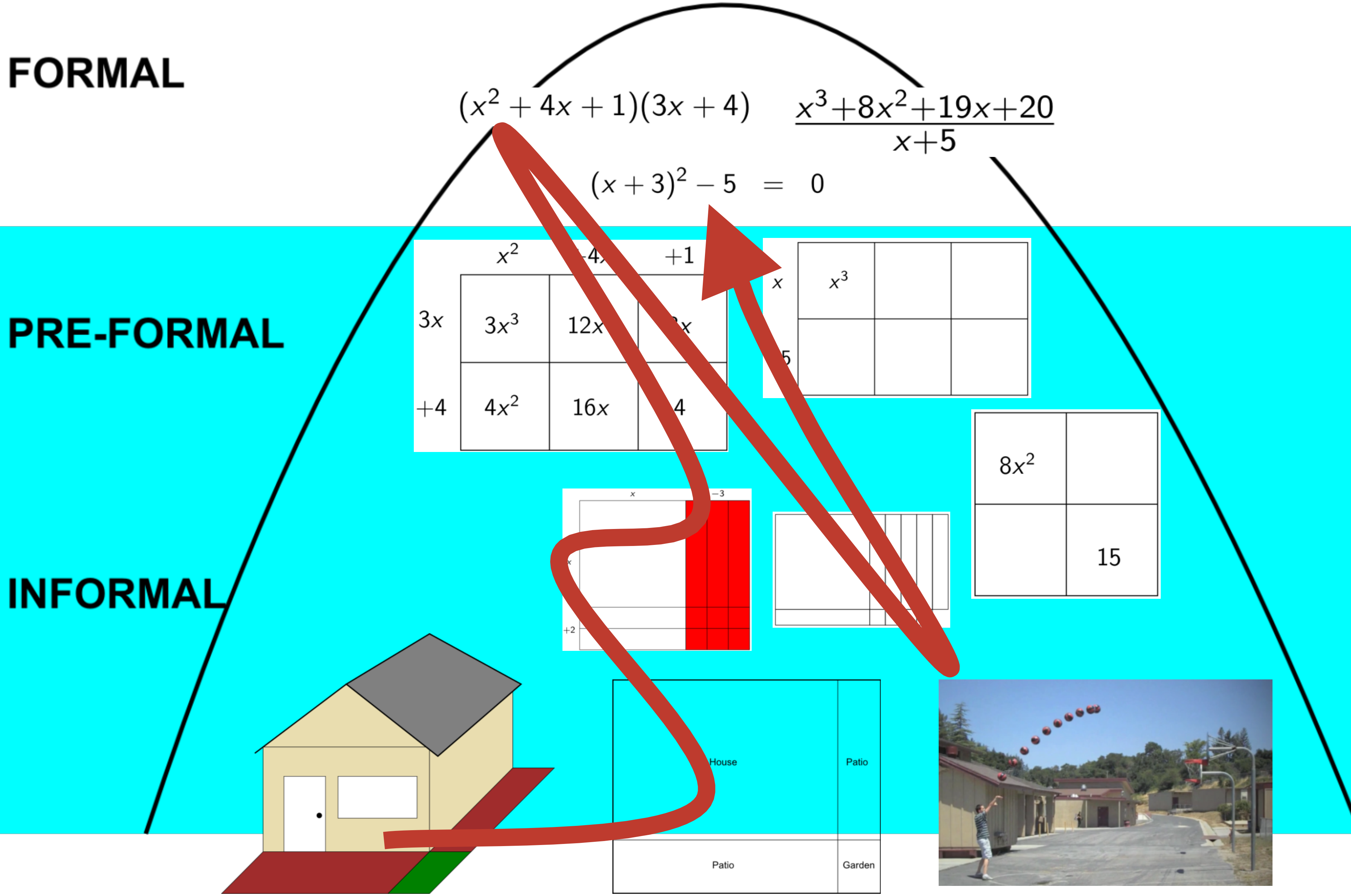
	x	$+3$	
x	x^2	$3x$	
$+3$	$3x$	9	-5

$$\begin{aligned}
(x+3)^2 - 5 &= 0 \\
(x+3)^2 &= 5 \\
\sqrt{(x+3)^2} &= \pm\sqrt{5} \\
x + 3 &= \pm\sqrt{5} \\
x &= -3 \pm \sqrt{5}
\end{aligned}$$

Completing the square problem string

1. $x^2 + 4x + 1$
2. $x^2 - 2x + 5$
3. $x^2 + 3x + 4$
4. $2x^2 + 8x + 2$
5. $2x^2 + 5x - 4$

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$\frac{x^3+8x^2+19x+20}{x+5}$
$(x+3)^2-5 = 0$
x^2
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$3x$
$3x^3$
$+4$
$4x^2$
$16x$
x
x^3
$8x^2$
15
x
-3
$+2$
House
Patio
Patio
Garden

Dividing polynomials

The last problem in multiplying polynomials was $(x^2 + 4x + 1)(3x + 4)$:

	x^2	$+4x$	$+1$
$3x$	$3x^3$	$12x^2$	$3x$
$+4$	$4x^2$	$16x$	4

Knowing the patterns of like terms, can you fill in what's missing if the product is $6x^3 + 17x^2 + 16x + 6$?

$6x^3$		$4x$
$9x^2$	$12x$	

What are the factors (dimensions) of the second box?

Knowing patterns of like terms, what's missing?

Knowing patterns of like terms, can you fill in what's missing if the product is $x^3 + 8x^2 + 19x + 20$?

x	x^3		
$+5$			

How is this different than asking students to divide $\frac{x^3+8x^2+19x+20}{x+5}$?

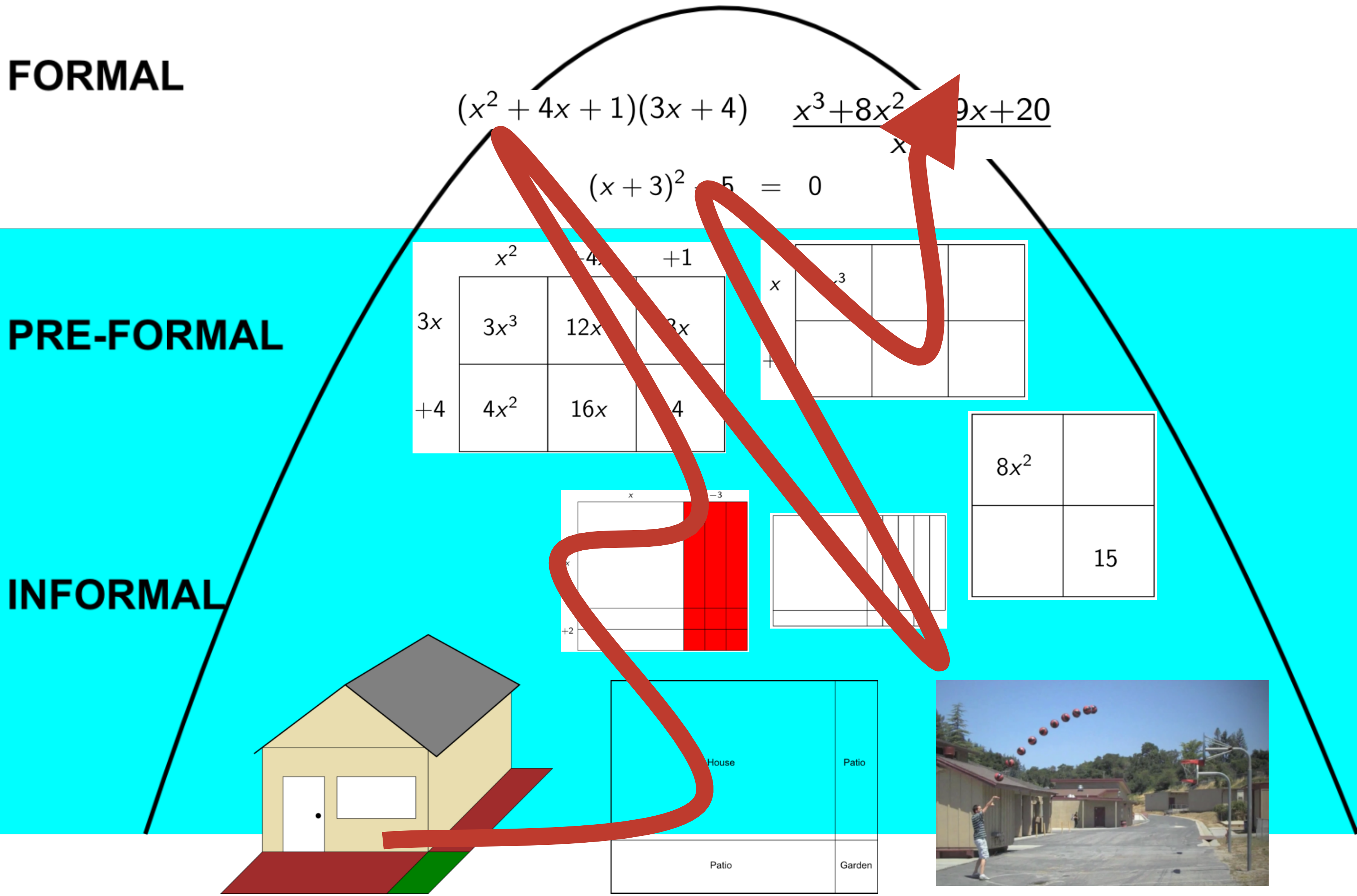
Dividing polynomials problem string

1. $$\frac{x^3 + 8x^2 + 23x + 24}{x + 3}$$

2. $$\frac{4x^3 + 11x^2 + 11x + 10}{x + 2}$$

3. $$\frac{3x^4 + 17x^3 + 10x^2 + x + 5}{x + 5}$$

4. $$\frac{x^2 + 4x + 6}{x + 5}$$


FORMAL
PRE-FORMAL
INFORMAL
$(x^2+4x+1)(3x+4)$
x^3+8x^2
$9x+20$
x
$(x+3)^2$
$= 0$
x^2
$+1$
$3x$
$+4$
$3x^3$
$4x^2$
$16x$
4
x
$8x^2$
15
x
-3
$+2$
House
Patio
Patio
Garden

Reflecting on modeling as organizing

Affordances of the area model

1. Developing understanding of polynomials as multiplicative objects
2. Flexible tool for algebraic manipulations

A procedural approach to polynomials

Multiplication: $(x + 5)(x + 3) = x^2 + 8x + 15 \Rightarrow$ FOIL

Factoring: $x^2 + 6x + 5 = (x + 5)(x + 1) \Rightarrow$ guess and check

Complete the square: $x^2 + 4x + 1 = (x + 2)^2 - 3 \Rightarrow$ algorithm

Polynomial division:

$$\frac{x^2 + 9x + 8}{x + 2} = (x + 7)(x + 2) - 6 \Rightarrow \text{long/synthetic division}$$

Design Time

Reflection questions to guide design

1. What is the **key mathematical structure** or idea?
2. What is a pre-formal **model** that can reveal this structure, and be used as a tool to do mathematics?
3. What **context** is (a) "begging to be organized" and (b) is such that a model *of* activity that can become a model *for* activity?
4. Sketch an **iceberg** that captures the experiential activity in context (lower level), the pre-formal model(s) (middle level), and the formal mathematics (upper level).

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