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Author(s): Frederick A. Peck

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The Intertwinement of Activity and Artifacts in Realistic Mathematics Education

Frederick A. Peck

Department of Mathematical Sciences, University of Montana

Frederick.Peck@UMontana.edu

Abstract

Realistic mathematics education (RME) is guided by the notion that mathematics is the human activity of mathematizing the world. In much of the RME literature, mathematizing is theorized to be an individual activity. In this paper I extend these ideas and discuss how mathematizing is intertwined with—mediated by, and distributed across—cultural artifacts. Such a cultural perspective on RME is a necessary consequence of RME's first principles, and has implications for many of RME's key principles. Exploring these implications is the next frontier in RME research.

Keywords: Realistic Mathematics Education, Sociocultural, Cultural psychology, Activity Theory, Algebra, Gesture, Distributed Cognition

The Intertwinement of Activity and Artifacts in Realistic Mathematics Education

Beginning in the late 1960s, the Dutch mathematician Hans Freudenthal (1968, 1983, 1987, 1991) began to sketch a vision for mathematics education based on the radical notion of mathematics as an *activity*, rather than a pre-existing structure or body of knowledge. This vision led Freudenthal to conclude that mathematics education should not be concerned with the transmission (or even the discovery) of mathematical structure, but rather with engaging students in the activity of structuring the world mathematically, which Freudenthal (1968, 1983) called "mathematizing." As Freudenthal and his colleagues worked out the implications of this vision of mathematics as the activity of mathematizing, they created a domain-specific instructional theory for mathematics known as Realistic Mathematics Education (RME; Gravemeijer, 1994a; Treffers, 1987, 1993; van den Heuvel-Panhuizen & Wijers, 2005). RME has always been a work in progress:

> RME started out as a vision, or as a philosophy of mathematics education that still had to be worked out. This is being done in developmental research projects, in which each time the research question is: What would mathematics education, which fulfills the initial points of departure, look like for a given topic? (Gravemeijer, 1999, p. 159)

This is the spirit in which I approach this paper. I respect the initial points of departure, while offering a new vision of RME based on cultural psychology. I am not the first to explore such concepts. Perhaps most prominently, Paul Cobb and colleagues (Cobb & Bowers, 1998;

Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Cobb, Zhao, & Visnovska, 2008; Cobb & Yackel, 1996; Cobb, 1998, 2002) explored the ways in which RME was compatible with cultural psychology. Ultimately, although these researchers found a "possible point of contact" (Cobb et al., 2008, p. 109) between RME and cultural psychology, they rejected the notion that RME ought to be considered a cultural theory.

I see this group's thought-provoking work as the start of a conversation amongst RME researchers. In the years since, this conversation has been taken up in mathematics education generally (e.g., McDonald, Le, Higgins, & Podmore, 2005; Radford, Bardini, & Sabena, 2007; Radford, 2008a, 2008c; Stevens & Hall, 1998) but it has not been taken up in the RME literature. In this paper, I aim to reignite the conversation in the RME community. I draw heavily on the groundwork laid by Cobb and colleagues, but ultimately will present a different conclusion that that of Cobb and his collaborators. I argue that a cultural perspective is a necessary consequence of the first principles of RME. I start by summarizing some key aspects of RME. I then discuss how some of these aspects are in tension with each other, and show how a cultural perspective resolves these tensions. I next turn to an empirical example to illustrate these ideas in a classroom. Then, I return to Cobb's research group, and re-interpret their reasons for rejecting a cultural perspective in light of my analysis. Finally, I conclude with the implications of my argument for RME.

1. Summary of RME, and a summary of cultural perspectives on learning

RME is rooted in Hans Freudenthal's notion that mathematics is the human activity of mathematizing the world. In 1987, Adrian Treffers (1987) described "five instructional principles" consistent with this approach. In the three decades since, these five principles have been adapted into six (van den Heuvel-Panhuizen & Wijers, 2005):

1. Activity principle: Mathematics is, first and foremost, the human activity of mathematizing. Mathematics education should involve students in mathematizing, and through this activity, students should create their "own [mathematical] productions" (Treffers, 1987, p. 249) as they engage in mathematical activity. That is, "mathematics can and should be learned on one's own authority and through one's own mental activities" (Gravemeijer, 2004, p. 109), such that students "come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible" (Gravemeijer, 1999, p. 158).
2. Reality principle: Freudenthal recognized that in order to be meaningful, mathematical activity had to be experientially real for students (Freudenthal, 1987). However, this did not mean that Freudenthal rejected formal and abstract mathematics. On the contrary, he observed that mathematicians use and discuss abstract mathematical productions as if they were real objects. Indeed, for the mathematician, these imaginary productions are real objects, and formal mathematics is experientially real. Learning mathematics should be a reality-expanding endeavor, such that "formal mathematics comes to the fore as a natural extension of the student's experiential reality" (Gravemeijer, 1999, p. 156).
3. Level principle (sometimes called "emergent modeling," Gravemeijer, 1999; or "progressive formalization," Webb, Boswinkel, & Dekker, 2008): Mathematical activity and productions exist at various levels of abstraction. At first, mathematizing takes place in a particular context, and productions exist as models *of* this activity in context. These models can themselves be mathematized into models *for* future, more general activity. Finally, at the level of formal activity, students use culturally-

accepted formal algorithms and means of formal symbolizing (Gravemeijer, 1999; Treffers, 1987; van den Heuvel-Panhuizen & Wijers, 2005; Webb et al., 2008).

4. Intertwinement principle: The various domains of mathematics should not be treated as silos, but rather intertwined. Students should engage in rich contextual activity that calls upon multiple domains at once (Treffers, 1987; van den Heuvel-Panhuizen & Wijers, 2005).
5. Interaction principle: Students should "share their strategies and inventions with each other" so that they can "get ideas for improving their strategies" (van den Heuvel-Panhuizen & Wijers, 2005, p. 290).
6. Guided reinvention principle: Learning is a process of reinvention (Freudenthal, 1991). The role of the teacher is to map a leaning route, "along which the students can find the intended mathematics for themselves… [and] students should be given the opportunity to build their own mathematical knowledge store on the basis of such a learning process" (Gravemeijer, 1999, p. 158).

As summarized above, there are two tacit aspects of RME in the six instructional principles. First, in RME mathematics is theorized as both an activity and a product (Gravemeijer & Terwel, 2000). Second, there is a prominent role attributed to the individual in RME: the individual is seen as the reinventor of mathematics and the possessor of private mathematical knowledge that results from this reinvention (see descriptions of Principles 1 and 6; Gravemeijer, 1999, 2004; Treffers, 1987). In the next section, I will show that these aspects are in tension with one-another and with the first principles of RME. To resolve this tension, I will draw on cultural theories of learning, often called sociocultural (Wertsch, 1994, 1998) or cultural-historical (Cole,

1996) psychology, or situated (Lave, 2008) or distributed cognition (Hutchins, 1995; Pea, 1993)[1]. In recruiting these theories, I will draw primarily on the notions of "culture" and "mediation," which I describe below.

I take a process and product view towards culture. Cultural *processes* are those that "accumulate partial solutions to frequently encountered problems" (Hutchins, 1995, pp. 354–355). The residua of these processes – the partial solutions themselves – exist in material and ideal form as cultural *artifacts*. These artifacts serve to propagate the achievements of past generations into the present. Further, the set of these artifacts constitutes culture-as-*product*: "the species-specific medium of human life" (Cole, 2010, p. 462).

Notice in the above quote that Cole refers to the "medium" constituted by culture-as-product. Human actions take place in a cultural milieu and as such they are *mediated* by culture. What this means is that human actions "involve not a direct action on the world but an indirect action, one that takes a bit of material matter used previously and incorporates it as an aspect of action" (Cole & Wertsch, 1996, p. 252). Here, the "bit of material matter" is a cultural artifact. Mediating artifacts do more than simply facilitate or amplify an action that would otherwise exist. Rather, they enable new forms of human actions, and further, they "act back" on the human actors, such that humans are constituted by culture. Thus artifacts and activity are *productively entangled*: In activity, humans produce artifacts as "partial solutions to frequently encountered problems," and these artifacts go on to play productive roles in future activity.

[1] These diverse traditions have different histories and make slightly different theoretical commitments (see, e.g., Nardi, 1996). However, one common thread is that all of the approaches reject dualist notions of the separation between mind and world, and pay particular attention to the meditational role of culture in human activity (O'Connor & Glenberg, 2003). As cultural mediation and non-dualism are my primary focus in this paper, I attend little to the distinctions between theories, and instead batch the theories together under the terms, "cultural perspective" or "cultural theories of learning."

This productive entanglement is easily seen in physical actions. For example, consider the activity of pole vaulting. Here, the pole is an artifact that exists as a partial solution to the problem of getting one's body over a high bar. It is a product of human activity. In addition, it mediates future activity. The mediating role of the pole is obvious: the pole-vaulter does not jump over the bar directly but rather she does so indirectly by incorporating the pole into the action. As such, the pole does not make humans better jumpers (i.e., amplifying an action that would otherwise exist), but rather it enables a completely new action that could not otherwise exist.[2] In addition, the pole acts back on the human actor: because it affords particular ways of being used, it shapes the actor in particular ways both in microgenetic and ontogenetic time. In microgenetic time, the pole shapes the movements of the pole-vaulter as she engages in the activity of pole vaulting. That is, the way she holds her hands and moves her body only makes sense if there is a pole. Across ontogenetic time, the pole shapes her body, callusing her hands in particular places and making certain muscles more prominent (Wertsch, 1998). It also shapes her identity: as she comes to know and use the pole fluently, she *becomes* a pole-vaulter (cf., Bowker & Star, 1999).

Cultural mediation plays a key role in psychological processes as well. Indeed, the key tenant of cultural psychology is that all higher psychological functions are culturally mediated (Cole & Wertsch, 1996; Vygotsky, 1978). What this means is that culture is not a background against which an independent mind creates knowledge through presumed biologically-universal

[2] The moment when an action becomes a "completely new" action is never clear, and is often contentious. For example, it is quite reasonable to say that pole vaulting is qualitatively different from jumping, or from any other human action that exists without a mediating artifact. But what about pole vaulting with a bamboo pole vs. a fiberglass pole? As Wertsch (1998) documents, pole vaulting with a fiberglass pole is very different from pole vaulting with a bamboo pole. Are they the same action? The answer was historically contested, with people setting new records with the fiberglass poles asserting *continuity* of activity, but critics arguing that it is in fact a *new form of action.* Similar struggles can be seen in mathematics education, for example, in the historical and contemporary struggles over the nature of the effect of calculators on school mathematics.

processes such as assimilation and accommodation (cf., Piaget, 1970). Rather, the human mind is constituted by and through culture. As described by Vygotsky (1997b, p. 138), "the central fact about our psychology is the fact of mediation."

A paradigmatic example psychological mediation comes from Vygotsky's double stimulation experiments (Vygotsky, 1978). The method involves giving a subject (in experimental conditions) a cognitive task to solve that is "beyond his [*sic*] present capabilities" (Vygotsky, 1978, p. 74). This task is the first stimulus. The experimental setting also contains a neutral object—a second stimulus:

> [F]requently we are able to observe how the neutral stimulus is drawn into the situation and takes on the function of a sign. Thus, the child actively incorporates these neutral objects into the task of problem solving. We might say that when difficulties arise, neutral stimuli take on the function of a sign and from that point on the operation's structure assumes an essentially different character (Vygotsky, 1978, p. 74).

Thus the second (neutral) stimulus becomes a mediating artifact as the subject creatively pours meaning into the neutral artifact and then uses the artifact to accomplish the task. Because the artifact fundamentally changes the character of the problem-solving operation, it is not sufficient to claim that the problem is solved by the individual acting alone. Rather, the artifact is a constituent part of the problem-solving activity, and therefore the problem is solved by a *system* that includes both persons and artifacts.

Cognitive processes, then, do not happen solely "in the head" (Cole & Wertsch, 1996, p. 253). Rather they are distributed—that is, "stretched over, not divided among" (Lave, 1988, p.

1)—persons and their cultural environment (Clark, 1998; Cole & Engeström, 1993; Cole, 1995; Hutchins, 1995). The wording in this quote is important. A common misunderstanding of so-called "distributed cognition" is that the claim is that cognition is something that can be partitioned such that separate partitions can be attributed to persons and artifacts respectively. What Lave's quote makes clear is that this quest is folly. Just as we cannot attribute different parts of walking to the right and left legs—walking only happens when the legs work together in a system, and we don't get a "half walk" if we take away one leg (cf., Rogoff, 2003 who makes a similar argument about "nature" and "nurture")—neither can we attribute different parts of cognitive activity to different elements of the person-culture system. Rather, persons and culture work together to create cognitive activity that is qualitatively different than the sum of the individual parts.

To recognize that artifacts are constituent parts of human cognitive activity does not reduce humans to simple stimulus-response creatures. This is because artifacts do more than simply store and transmit meaning. As the example of double stimulation makes clear, human creativity and agency are fundamental components of mediated activity. For Vygotsky, human agency is always mediated by artifacts: we control ourselves through artifacts of our own creation (Engeström, 2007; Wertsch & Rupert, 1993), and become particular kinds of people as we work with artifacts (Bowker & Star, 1999). This is how culture "acts back" and constitutes the actor.

To sum up, a cultural perspective puts culture "in the middle" (Cole, 1996, p. 116) of all human activity. Activity and culture become intertwined: humans produce culture, in the form of artifacts, as they engage in activity, and those artifacts mediate and transform future activity. In the process, culture "acts back" and produces people in particular ways as they engage in activity. Thus, a cultural perspective is non-dualist and relational. People, culture, and activity can only be understood in relation to each other. In the next section, I will show how this cultural perspective is not only compatible with RME, it is a necessary consequence of the three key aspects of RME that I highlighted above.

2. A cultural perspective is a necessary consequence of the first principles of RME

I will make this claim in two parts. First, beginning with the first principles of RME, I will show that mathematizing is a mediated activity. Second, I will show that mathematical productions are cultural artifacts. The upshot of these two claims is the following larger claim: Mathematizing is always a culturally mediated activity and mathematics is a cultural artifact.

Claim 1: Mathematizing is a mediated activity

Let's start from the initial point of departure in RME, Freudenthal's adage that mathematics is the human activity of mathematizing—that is, structuring the world mathematically. To warrant the claim that this is a mediated activity, I will first make a softer claim, that mathematizing is *sometimes* a mediated activity. One example is sufficient to make this softer claim. The example that I will use comes from Jean Lave and colleagues (Lave, 1988), and involves a weight-watcher in his kitchen trying to figure out how to make a serving of cottage cheese that is ¾ of 2/3 of a cup. Rather than multiplying 2/3 cups by ¾, the weight-watcher uses a measuring cup to measure out 2/3 of a cup of cottage cheese, pours this measured serving on the table, shapes it into a circle, cuts the circle into quarters by making perpendicular radial cuts with his finger, and then scoops one of these quarters back in to the cottage cheese container. He is left with ¾ of 2/3 of a cup.

Comparing the end-state (a precise measurement of cottage cheese, shaped into a ¾ circle) with the beginning state (a container of unmeasured, shapeless cottage cheese) it's clear that the end is more mathematically-structured than was the beginning, and thus the weight watcher *mathematized* (i.e., he structured his environment mathematically). Examining the *activity* of mathematization, it is clear that this was a mediated activity—at the very least, it was mediated by the measuring cup in the same way that the pole mediates pole vaulting.

Thus mathematizing is—at least sometimes—a mediated activity. To make the stronger claim, that mathematizing is *always* a mediated activity, I remind the reader that within RME, mathematics is an activity *and* a product. In itself, this is not a controversial statement within RME. Even though Freudenthal critiqued educational practices that focused on "the transmission of mathematics as a pre-formed system" (Gravemeijer & Terwel, 2000, p. 779), he also recognized that mathematics does exist as a product. For example, in his *China Lectures* (Freudenthal, 1991, p. 16), he argued that mathematics is a human activity, but clarified, "if I were to continue in the same way, that is, by focusing on the mathematical process, I would imprudently be neglecting the medium in which this process takes place." For Freudenthal, this "medium" is the world of mathematical productions.

The existence of mathematics-as-product is essential to understanding Freudenthal's notion of mathematizing: "a form of organizing that also incorporate[s] mathematical matter" (Gravemeijer & Terwel, 2000, p. 781). Note the similarity between these quotes about mathematics, and the definitions of culture and mediation given earlier. Below, I will have more to say about the relationship between mathematics and culture. For now, it is sufficient to note that if mediation involves the incorporation of pre-existing matter into activity, then mathematizing is *always* a mediated activity. Indeed, the mediation of mathematics-as-product is what separates mathematical activity (i.e., mathematizing) from non-mathematical activity.

We can see this in the weight-watcher example. The reason that I could claim that the weight watcher was structuring his world *mathematically* is because his activity was mediated by mathematical productions. As one example of many, consider how the mathematical production of *circle* mediates the activity. On the one hand, we can say that the Weight Watcher uses "circle" because he *circlitizes* the cottage cheese. On the other hand, we can say that "circle" uses the Weight Watcher, because "circle" structures his hand movements in a particular way. Just as the pole acts back on the pole vaulter, so too does "circle" act-back on the Weight Watcher in microgenetic time. The activity of structuring the cottage cheese mathematically is stretched across the Weight Watcher and mathematical productions, and indeed all mathematizing has this character.

Thus I have warranted my first claim, that mathematizing is a mediated activity. It is mediated by material artifacts in the environment and by mathematical productions that constitute the "medium" in which the activity takes place. But just what are mathematical productions? Within RME, they are commonly referred to using individualistic terms such as "mental objects" (Freudenthal, 1983, p. 33), and "private knowledge" (Gravemeijer, 1999, p. 158). Below I argue that, in contrast to these individual-centric notions, mathematical productions are cultural artifacts.

Claim 2: Mathematical productions are cultural artifacts.

To make this claim, I will draw on the reinvention principle. As described by Gravemeijer (Gravemeijer, 2004, p. 114), this means that "students should be given the opportunity to experience a process similar to the process by which a given piece of mathematics was invented." There are two notable features of this description. First, the notion that mathematical productions are invented rather than discovered suggests that they are the products of human activity as opposed to being a part of a Platonic structure that exists independent of humans. Freudenthal (1968, p. 6) also located the origins of mathematics-as-product in human activity, stating, “arithmetic and geometry have sprung from mathematizing part of reality.”

Second, the “re” in reinvention along with the past tense in Gravemeijer’s phrase “a given piece of mathematics was invented” communicates that the productions that a given student is to reinvent pre-exist the student. This is not simply a linguistic accident. Freudenthal was very careful in his choice of the term “guided reinvention,” as opposed to “construction” or “discovery” or even “invention”:

> Guiding reinvention means striking a subtle balance between the freedom of inventing and the force of guiding (…) [T]he learner’s free choice is already restricted by the “re” of “reinvention.” The learner shall invent something that is new to him [*sic*] but well-known to the guide (Freudenthal, 1991, p. 48).

None of this is controversial within RME. But once we understand that productions pre-exist individual students, it becomes impossible to maintain the notion that productions are private and internal. So just what is the ontological nature of mathematical productions? As documented above, the notion of guided reinvention imposes strict criteria on the ontology of mathematical productions within RME: First, productions are the product of human activity so we have to reject the idea that they exist in some metaphysical Platonic structure, independent of humans. Second, they are durable and shared, existing across time and outside of any single individual. This means we have to reject the idea that productions are individual possessions.

The only object that meets these criteria is a cultural artifact. Thus, this warrants my claim that mathematical productions are cultural artifacts (see also, Peck & Matassa, in press).[3]

Summary before moving on

Thus far I have discussed how, within RME, mathematics has been theorized as a collection of "mental objects" and learning has been taken up as the acquisition of "private knowledge." I have demonstrated that such a position is in tension with the first principles of RME (specifically the activity principle and the guided reinvention principle). I have shown that instead, a cultural perspective is a necessary consequence of these principles. Specifically I have claimed that, if one follows the first principles of RME, mathematical productions must be considered cultural artifacts, and that mathematizing must to be understood as a culturally-mediated activity. In other words, culture—in the form of mediating artifacts that are produced through, and recruited into, activity—is at the very heart of RME. To illustrate this idea in practice, I next turn to an empirical example.

3. Mathematizing as a mediated activity: Two empirical examples

The example for this paper comes from a design study (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003; Gravemeijer & Cobb, 2006) in an activity-based Algebra I classroom in which I was the teacher (Peck, 2015b). Throughout the school year, my colleagues and I

[3] Following the definition of culture given earlier, when I say that mathematical productions are *cultural artifacts*, I mean that they are the collected product of human mathematical history, a "symbolic and materially constituted social inheritance" (Cole, 2010, p. 462). In this way, mathematical productions exist in the same way that languages exist, or symphonies or literature.

To say that these artifacts pre-exist a given individual is to recognize that "we live from birth to death in a world of persons and things which in large measure is what it is because of what has been done and transmitted from previous human activities" (Dewey, 1938, pp. 39–40). However, this is not to say that all humans are born *aware* of language, symphonies, literature, or mathematics, nor is it to suggest that the cultural world into which we are born is static. As cultural objects, mathematical productions can be *objectified* (Radford, 2008b, 2010, 2013; Roth & Radford, 2011) through social processes such that they become real to individuals. Further, they can be manipulated and extended by humans to create new culture. That is, just as Shakespeare manipulated existing language to create new works, so too can humans manipulate existing mathematics to produce new mathematics.

designed activities and activity sequences using RME principles, and RME guided the design of the study under consideration in this paper. For this study, our goal was to create a local instructional theory (Gravemeijer & Cobb, 2006; Gravemeijer & van Eerde, 2009; Gravemeijer, 1994b) for how students could be guided to reinvent and connect five sub-constructs of slope in a high-school Algebra I course. The five sub-constructs of slope that we focused on were: (1) Rate of change; (2) the parametric coefficient (the a in $y = ax + b$); (3) the algebraic ratio ($\frac{y_2-y_1}{x_2-x_1}$); (4) the geometric ratio ("rise over run"); and (5) the physical property (steepness) (Stump, 1999).

In this paper, I will focus on a single problem-solving episode, in which a group of students reinvents the algebraic ratio. I focus on reinvention because it is perhaps the quintessential mathematical activity in RME. As I discussed in Section 1, reinvention is mathematization, and it is also the mechanism through which learning is hypothesized to occur in RME (Freudenthal, 1991).

Prior to the episode in this paper, the class had reinvented and/or made meaningful two of the sub-constructs of slope: rate of change and the parametric coefficient. As I document elsewhere (Peck, in progress), the class understood rate of change as a both a measure of an intensive quantity (see below) and a measure of linear covariation between two quantities, which can be iterated and accumulated via multiplication to make predictions. Students coordinated this latter understanding of rates of change with algebraic equations to reinvent the parametric coefficient..

In addition, the class had reinvented and/or used a number of other mathematical productions in class, some of which became consequential in the problem-solving episode. In particular, ratio tables (Middleton & van den Heuvel-Panhuizen, 1995), fractions-as-quotients (Kieren, 1980), the "find one" strategy (Peck & Matassa, in press), and a unit rate strategy

(Cramer, Bezuk, & Behr, 1989). The first three of these productions emerged in the classroom as "partial solutions to frequently encountered problems" before the design study in a unit that involved finding unit values given a many-to-many relationship (the sort of problems that a mathematician might classify as involving partitive division). For example, as shown in Figure 1, students solved equal-sharing problems using equipartitioning (Empson, 1999; Streefland, 1993; Wilson, Edgington, Nguyen, Pescosolido, & Confrey, 2011), as well as other missing value proportional reasoning problems (Kaput & West, 1994).

Figure 1 shows an example of the first three mathematical productions above. On the left side, the student has constructed a ratio table, which she coordinated with the "find one" strategy to find the equal share. The "find one" strategy was invented and named by the class during the course of this early unit. It links the division operation to situations in which the goal is to find the value of one object. The student's use of this strategy is indicated by the downward curving arrows coordinated with the division operation, which together link the initial situation on the first row of the ratio table to the equal share on the second row of the ratio table. The student expresses the equal share using a fraction-as-quotient ($2 \div 5 = 2/5$). Her understanding of the link between the fraction and the sharing activity is shown in the diagram on the right side, which depicts the operation of sharing using equipartioning and distributing (see Peck and Matassa, in press for a detailed description of how these artifacts emerged in the classroom as "partial solutions to frequently encountered problems").

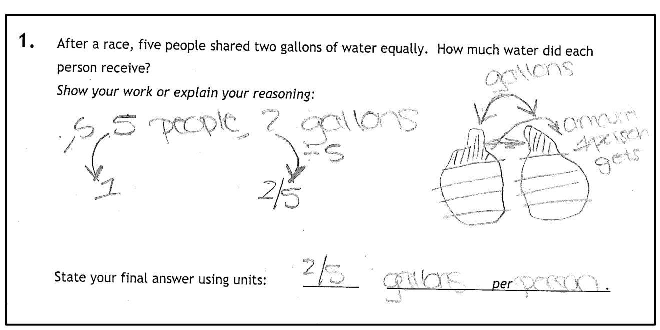


Figure 1: Student work showing ratio tables, the find one strategy, and fractions-as-quotients.

While the "find one" strategy is sufficient for missing value proportional reasoning problems in which the missing value corresponds to a unit value in the given variable (e.g., the amount of water for *one* person), it will not suffice for problems in which the missing value corresponds to a non-unit value in the given variable (e.g., how much water for three people?). For problems such as this, students used a *unit rate strategy*, which involves using the "find one" strategy to create a unit rate given a many-to-many relationship, and then scaling that rate using multiplication to make a new many-to-many relationship (Cramer et al., 1989). Notice that, while this strategy is more general than the "find one" strategy, it is still limited to situations in which the two variables are proportional to each other. For more general linear situations, in which changes between variables are proportional but values may not be, the unit rate strategy is not sufficient to make predictions. For these situations, a more general strategy involves subtracting values of the independent and dependent variables to find changes in each, and then dividing these the latter by the former to create a unit rate. This is the strategy depicted by the algebraic ratio, $(y_2 - y_1)/(x_2 - x_1)$.

Thus, to guide students to reinvent the algebraic ratio, we presented them with linear, but not proportional, situations and asked student to make predictions given two value pairs. The class' first experience with such a situation was the "window problem" shown in Figure 2. As shown, the window problem asks students to make a prediction in a functional situation in which the dependent variable (price) varies linearly, but not proportionally, with the independent variable (number of windows). As will become clear, the students in the focal group do not have a strategy to solve such a problem in the beginning of the episode, but by the end they have created the "subtract and divide" strategy depicted by the algebraic ratio. Thus it is appropriate to refer to their activity as "reinvention."

A. B.

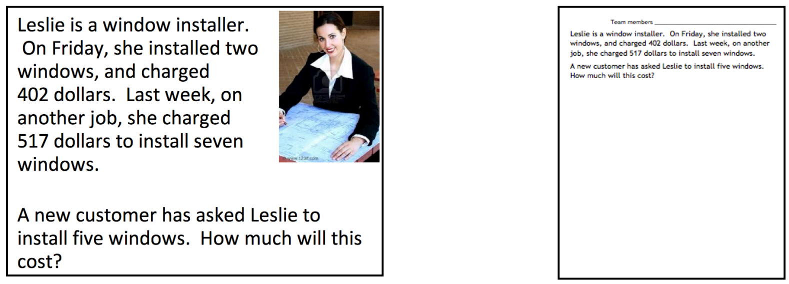


Figure 2: The window problem as it is represented on the white board (A) and on the group's paper (B).

To trace the process of reinvention, I will analyze the problem-solving activity of four students (referred to using the pseudonyms David, Stacy, Melissa, and Tyler) as they work on the window problem. As shown in Figure 3, the students are seated in a cluster of four individual desks, with David and Stacy facing Tyler and Melissa. Melissa has a large piece of paper in front of her with the problem written at the top. The problem is also projected onto a whiteboard that is located behind Melissa and Tyler and in sight of David and Stacy. The physical layout of these inscriptions will become consequential, thus Figure 2 shows both inscriptions.

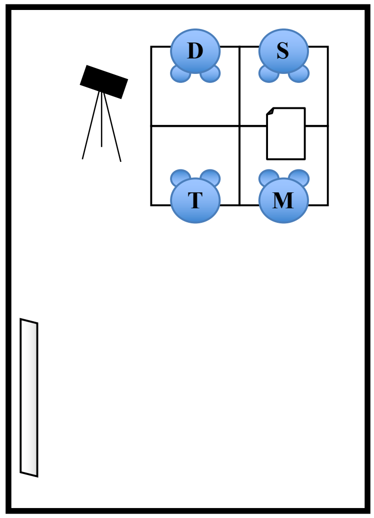


Figure 3: A schematic (not-to-scale) of the classroom, highlighting the position of the students in the focal group, as well as the position of the group's shared paper, the whiteboard (lower right) and the camera, relative to the focal group.

I used ethnographic microanalysis (Erickson, 1992, 1995) to analyze the moment-by-moment multi-modal mathematical activity of this group. As recommended by Erickson (1992) and others (Arzarello, Paola, Robutti, & Sabena, 2008), I collected video and audio recordings to facilitate the microanalysis. The video camera was positioned such that the faces, hands, and desktops of all four students were generally visible in the screen at the same time. A wide-angle microphone, placed in the center of the group, captured audio. The group produced a single inscription on their poster-sized piece of paper, which I collected. The group also made use of the projected inscription of the problem statement. This projected inscription is off camera. However, I have the PowerPoint slide that was projected (Figure 2A), as well as a diagram of the classroom that shows how students were positioned relative to the white board (depicted in Figure 3).

My unit of analysis is the *semiotic bundle* (Arzarello et al., 2008; Arzarello, 2006), which can be explained as follows: As students engage in mathematical activity, they marshal multiple *semiotic means of objectification* (Radford, 2003), including talk, gesture, body position, gaze, inscriptions, and other artifacts. The semiotic bundle is the coordinated ensemble of these semiotic means that students recruit within and across moments of activity. For example, in the episode that follows, I will show how Stacy coordinated talk, the group's written inscription, and multiple mathematical artifacts in a key turn at talk that created the need to reinvent the algebraic ratio. Notice that this bundle includes cultural products such as inscriptions and mathematical artifacts. Thus, using a semiotic bundle as my unit of analysis is appropriate for my task of showing how mathematical activity is distributed across cultural artifacts.

I analyzed these bundles synchronically and diachronically. Synchronic analysis examines how the semiotic resources are coordinated in a single moment. Diachronic analysis explores how the semiotic resources change over time (Arzarello et al., 2008; Arzarello, 2006). For example, I will show how David employed talk, gesture, and the projected inscription to map *values* of the numeric words that were projected onto the physical *space* that these words occupied in the inscription (synchronic analysis). I will also show how this gesture itself became an artifact for the group across time, even as it was coordinated with different inscriptions and talk (diachronic analysis).

To facilitate analysis, I created multiple data sources. First, I created a video that places the group video captured by the camera alongside a scanned copy of the students' poster. This is important because the poster is generally too small to read in the overview video. Placing the two side-by-side helps to reveal the coordination between talk, gesture, and the inscription. From this video, I created a transcript of talk and gesture, using a transcription style based on the format described by Ochs (1999) for capturing both verbal and non-verbal behavior. Specifically, this means that I have two separate columns in my transcript: one for talk and one for non-verbal action.

Reinvention as a mediated activity

The episode begins with Melissa reading the problem out loud. Immediately after Melissa reads the problem, two things happen. First, the group begins to solve the problem by recruiting mediating artifacts:

Segment 1: Recruiting artifacts

13	S: A new customer asks[1] (.) Leslie[2] to install five windows how much is ( )[3] So what we should do is [set it up	(([1]D looks down towards paper)) (([2]D looks up at screen)) (([3]S looks toward paper, D continues to look toward screen))
14	M: [is do tw- divided by two[1] divided by two[2] to find the one, the price of one window?	(([1]M points at the "2" on paper)) (([2]M points at the "402" on paper))

As will become apparent soon (see Figure 5 below), Stacy recruits a ratio table into the activity when she calls for the group to "set it up" in turn 13. In turn 14, Melissa recruits a find-one strategy.

Second, the group fractures into two ensembles of persons and artifacts. I have analyzed the social dynamics that lead to, and resulted from, this fracturing elsewhere (Peck, 2015a). For the analysis and arguments presented here, the primary focus is the nature of the ensembles themselves. As shown in Figure 4, one ensemble consists of Stacy, Melissa, Tyler, and the group's paper. The other ensemble consists of David and the white board.

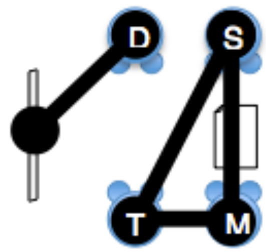


Figure 4. The social organization after 13 turns at talk.

The intra-ensemble interactions are exemplified in Segment 2 below:

Segment 2: Two kinds of interactions

31	S: This rate is four oh two over two	
32	M: Divide first [( )	
33	S: [That's the rate, yeah	
34-36	((Crosstalk))	
37	S: [And then equals=	((D raises arm))
38	M: [Five seventeen over seven	((M writes on paper))
→39	S: =equals on:e[1] (.) window. And then[2] also four oh two[3] (...)[4] over two equals one window.	(([1]D moves fingers rhythmically)) (([2]D points to board)) (([3]D moves fingers rhythmically)) (([4]D makes "gun" gesture))

Note the different sorts of interactions that are happening within the triad (Melissa, Stacy, Tyler, and the group's shared paper) on the one hand, and David and the white board on the other. Melissa and Stacy talk about the problem using overlapping and latched speech. Their turns build on each other, and they coordinate their talk with their shared artifact. They are continuing the strategy that Melissa and Stacy set out in Segment 1, using a find-one strategy to find the rate associated with each of the two (window, cost) pairs given in the problem.

In Segment 2, turn 39 (arrowed), David is doing something very different. He moves his fingers rhythmically. He points to the board. His movements are not coordinated with the talk or work of the triad.

Over the next thirteen minutes, David continues to gesture at the white board. We'll return to David in a moment. For now, let's follow the work of the triad. Over these 13 minutes, the triad produces the inscription shown in Figure 5. This inscription coordinates ratio tables, the find one strategy, fractions-as-quotients, and the unit rate strategy into two material assemblies, one for each of the (window, cost) pairs in the problem. One assembly is in the lower left of the figure, and the other is in the upper right. Each assembly can be read as a 2x2 vertically-aligned ratio table (similar to the ratio table depicted in Figure 1), with columns separated by the equals sign. Each column contains like-units (cost in the left column and windows in the right column), and each row contains a ratio of cost to windows. At the lower left of each assembly is the prediction for five windows, which the triad arrived at via a unit rate strategy (recall from above that a unit rate strategy involves finding a unit rate via the find one strategy—which occupies the first row of each assembly—and then scaling that unit rate to make a prediction—in this case, shown via downward curving arrows coordinated with a multiplication operation).

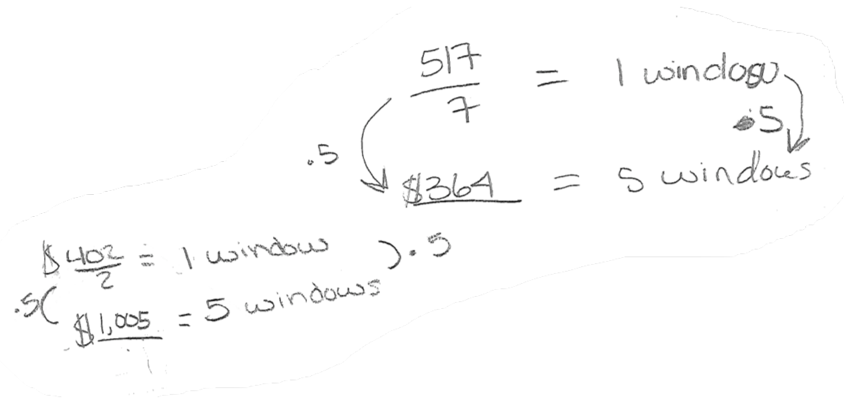


Figure 5. The triad created two coordinated assemblies of artifacts

Thus far, the triad's problem-solving process has been mediated by multiple mathematical artifacts: rates of change, fractions-as-quotients, ratio tables, the find-one strategy, and the unit rate strategy. That the students are engaged in mathematical structuring activity (mathematizing) should be obvious: the material assemblies that the triad has produced by bringing the various artifacts into coordination with each other and with the problem statement constitute a more-structured state of the problem context than that with which they began.

But the structuring activity is not located entirely within or even between the students in the triad. The artifacts too, structured the activity. For example, the "find one" strategy structured much of the triad's activity. Earlier, I stated that Melissa recruited the find one strategy into the activity in turn 14. This is accurate. But it is also accurate to say that the find one strategy recruited the triad into doing a particular kind of activity. Indeed, the triad mentions the need to "find one" using the find one strategy eight separate times during the first thirteen minutes. Similarly, the triad uses ratio tables to structure the problem, but also, the ratio table uses the students to structure their activity around the particular organization that the ratio table demands. This can be seen, for example, in the "blanks" drawn as horizontal underlines in the lower left corner of each assembly in Figure 5. Initially, these “blanks” were empty, and they were only later filled in by the students as they completed the arithmetic to make each prediction. Drawing the “blank,” then, is an action demanded by the ratio table to maintain its row and column organization. Without the ratio table, such an action is nonsensical. Thus, the students use artifacts and the artifacts use the students, such that the students’ mathematizing activity is distributed—stretched over, not divided among—the artifacts that co-constitute it.

Up to now, the triad’s talk has looked very similar to that depicted in Segment 2. The students in the triad build on each other’s turns, they are hunched over the shared paper, and they maintain a solution-oriented affect. In other words, they are proceeding as if they are comfortable that they know how to solve the problem. After the production of each assembly, however, things change. Stacy exclaims, “what the heck?” (turn 181) and Tyler moans, “Oh God!” (turn 194). We uncover the source of their displeasure when Stacy asks, in turn 198, “are these supposed to, like, relate?”

Stacy’s use of the deictic, “these” is a reference to the two predictions, and in asking if they are “supposed to like, relate,” Stacy’s talk works to bring the two predictions into coordination. Stacy’s question is a sense-making question—a *semiotic* question—that marshals together the bundle of semiotic resources the group has assembled so far. The distress evident in Stacy and Tyler’s earlier exclamations suggests that something in this bundle doesn’t conform to their expectations.

The trouble, of course, is that the forms of activity that are enabled by the artifacts that the students have recruited are not sufficient to solve this this problem. As co-constituents of the triad’s activity, the artifacts enabled certain forms of activity, but they also constrained that activity (cf., Pickering, 1995). In the window problem, these constraints manifest themselves as two different predictions for what ought to be the same price. The students' exclamations, and Stacy's question, are recognitions of the contradiction produced by this constraint. For the triad, the problem of finding the price of five windows has now become a dilemma of two different predictions. The students become visibly and audibly frustrated.

What we have, then, is a task that is, to borrow Vygotsky's phrase, "beyond the present capabilities" of the triad and the artifacts that help to constitute their possibilities for action. But even as human activities are enabled and constrained by artifacts, humans maintain a very particular kind of agency: the potential for "intentional collective and individual actions aimed at *transforming* the activity" (Engeström, 2007, p. 381, italics added; see also Kaptelinin & Nardi, 2005, Chapter 10). This transformative agency, like all human agency, is exerted through artifacts.

Here we rejoin David. Recall that when we left David, he was gesturing at the white board. For the past 13 minutes, his activity has largely considered of such gestures. At times, he makes bids to rejoin the group, but up until now, those bids have been denied. Now, however, the triad is in a very different state than they had been up to this point. In Segment 3 below, David introduces the key insight needed for the group to reinvent the algebraic ratio. As we will see, this insight is itself distributed across artifacts.

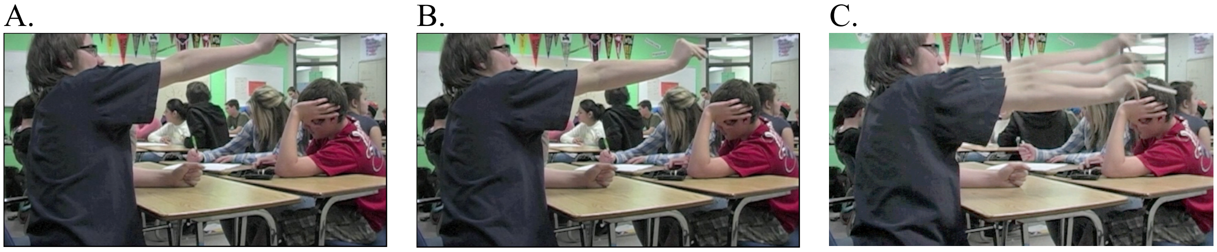
Segment 3: A key insight

253	D: What's uh, what's a hundred and fifteen divided by	((D shifts gaze back and forth between board and center of group))
254	S: Where are you getting a hundred and fifteen?	

→255	D: Four hundred and two[1] plus a hundred and fifteen is five hundred seventeen[2]. So that's the price between the two[3], so if we divide that by[4]::, [that's what I'm trying to figure out you'll get the price of one	(([1]D points to board)) (([2]D lowers hand slightly, still pointing to board)) (([3]D moves hand up and down rapidly)) (([4]D Lowers arm, turns towards group))
256	M: [So between here[1] and here[2] is a [hundred,	(([1]M points to paper with pencil)) (([2]M moves pencil slightly higher on paper))
257	S: [Oh	
258	M: it's a [hundred and fifteen dollars	((M moves pencil back and forth))

David initiates the event with a question (turn 253). Stacy's uptake, in the form of another question, works to position David as a contributing member of the group, at least temporarily. In turn 255 (arrowed), David explains his strategy. As shown in Figure 6 this explanation involves the coordination of talk and gesture.

A. B. C.



A. Four hundred and two plus a

B. hundred and fifteen is five hundred seventeen.

C. So that's the price between the two.

Figure 6. David's gesture in turn 255

On first glance, the gestures shown in Figures 6A and 6B seem to be deictic. As David says the words "four hundred and two," he points to the number 402 on the white board (Figure 6A), and similarly for "five hundred seventeen" (Figure 6B; recall from Figure 2 that the problem statement is written such that the numbers are on different lines, with 402 above 517). These deictic gestures bring the problem statement into coordination with David's speech and in this respect they facilitate the problem statement being used as a warrant for David's use of the numbers 402 and 517 (notice that whereas David was questioned about where the number 115 came from [turn 254], no one questions him on where he got the numbers 402 and 517).

However, these gestures do more than simply index the problem statement to provide a warrant for a claim. In addition, they coordinate the written numerals with physical locations in space, enabling a phenomenon that Wittgenstein (1958) refers to as *seeing-as.* In this case the gestures facilitate *seeing* the written numerals *as* locations in space. Melissa's uptake in turn 256 supports this claim. Notice that as she mimics the gesture (more on this below) she uses the word "here" while pointing to numbers on the paper. The word "here" is a spatial pronoun, whose antecedents are points in space and not values of numbers. Thus, Melissa's use of the word "here" to refer to numbers on a paper indicates that she is seeing the numbers on the paper as points in space.

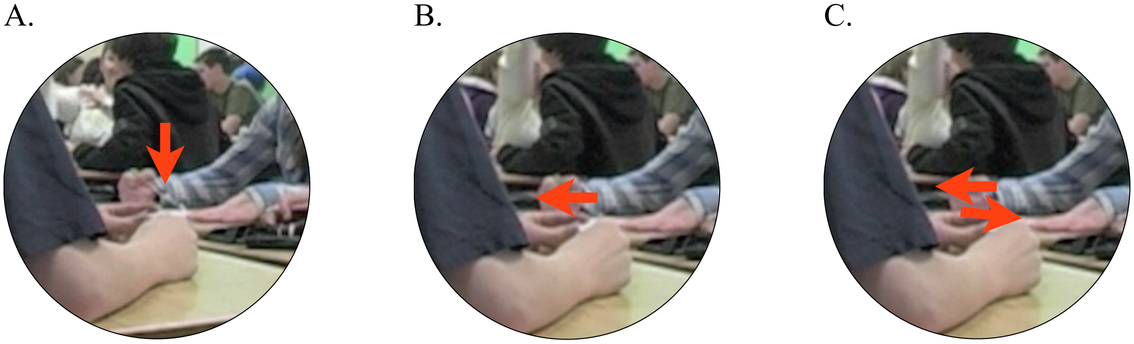
Now let's return to turn 255 and consider David's next move, shown in Figure 6C. In this move, David says, "so that's the price between the two" while he moves his hand rapidly up and down between the locations that he pointed to in Figures 6A and 6B. Like the gestures in Figures 6A and 6B, this gesture does both deictic and creative work. As a deictic, it works to provide a referent for the pronoun, "the two." As a creative act, it works in coordination with David's talk to make the physical space between the written numerals meaningful.

First, consider the talk: David says, "the price between the two." The word "between" is a spatial word. It is sometimes used to refer to an intermediate *point*, and sometimes to an intermediate *space*. The former is the most likely use in the context of prices (as in, "the price for five windows ought to be between the price for four windows and the price for seven windows”). However, that isn’t how the word is used here – David uses “the price between the two” to refer to the number 115, which is not an intermediate price between 402 and 517. To say that 115 is “the price between” 402 and 517 is to use “between” to refer to intermediate *space*. Of course, talk is not acting alone here, it is coupled with gesture. In this case, the gesture also works to put meaning into the physical space between the two written numerals. This is because the rapid movement in the gesture draws the eye to the space between the two endpoints rather than a specific intermediate point.

Overall, in turn 255 David’s talk and gesture work together to map the the physical *space* occupied by the written numerals on the whiteboard with the *value* of those numbers. Talk and gesture then work together to make the physical space between the written numerals meaningful. The overall effect is that physical space can be *seen as* numerical values. To be sure, these moves are afforded by the way that the numerals occupy space on the white board (if the problem statement had been verbalized instead of written, this affordance would have been absent), as well as by cultural technologies that facilitate number-as-space mappings, such as the “mental number line” (Núñez, 2011). However, just because artifacts afforded this way of seeing does not mean that it was “inside” the whiteboard *a priori*. It certainly wasn’t designed that way; when we created the slide that was projected onto the whiteboard (Figure 2A) we never considered how the spatial locations of the numerals might be meaningful. If the numerals were separated in physical space it was only because this is a consequence of written language. So where did this meaning come from? It came from David, from the whiteboard, and from other cultural technologies such as the mental number line. It is a distributed accomplishment, achieved through the coupling of gesture and talk.

But the gesture does even more. In addition to facilitating *seeing as*, the gesture itself becomes a representational artifact, signifying the white board. This can be seen in turn 256 (Figure 7), where Melissa revoices David's talk and mimics the gesture on the group's paper. In doing so, she brings the paper into coordination with the white board. This is key because David's strategy is linked to the white board—in particular to the mapping between the physical location of the numbers on the white board and the value of those numbers. By coordinating the paper and the white board, Melissa mobilizes the strategy *into* the triad. So mobilized, the strategy can be taken up by other members of the group, which Tyler and Stacy do in turns 260 and 261, respectively. The gesture does the work of mobilization. In creating the gesture, David created the rails (Latour, 1983) on which the strategy travels.

A. B. C.



A. So between here

B. and here

C. is a hundred and fif- it's a hundred and fifteen dollars

Figure 7. Melissa's gesture

The whiteboard, then, began as a neutral object. As David interacted with it, he imbued it with meaning, and from then on the problem-solving episode took on a different character—the group repaired its fracture, and then went on to reinvent the algebraic ratio and solve the problem

(Peck, 2015a). The white board thus served as a second stimulus, which transformed the group's activity and mediated the group's reinvention of the algebraic ratio. In this group, reinvention is a mediated activity—a sensuous, extra-psychological phenomenon that is distributed across persons and artifacts.

4. Returning to Cobb et al.'s argument

Thus far, I have presented and exemplified my argument that a cultural perspective is a necessary consequence of the first principles of RME. As I said in the introduction, I am not the first person to consider such a position. Most prominently, Paul Cobb and his collaborators considered this idea in the 1990s and early 2000s. However, Cobb and colleagues came to a different conclusion than I have. In this section, I return to their argument, and re-interpret it in light of the analysis that I have presented here. In doing so, I what to be clear that from my perspective Cobb and colleagues have done more to push RME towards a cultural perspective than anyone, and their groundbreaking work is what makes this paper possible. This section should be read as a reinterpretation, rather than a refutation, of their important work.

Recall that Cobb and colleagues (2008, p. 109) noted a "possible point of contact" between RME and cultural psychology concerning the role of mediating artifacts in both RME and cultural theories of learning. Often they stress the importance of mediating artifacts to both cultural theories of learning and RME as more than simply a "possible point of contact." For example, in a paper in which Cobb and colleagues describe an RME instructional sequence in a first-grade classroom, they "fully accept Vygotsky's fundamental insight that semiotic mediation is crucially involved in student conceptual development" (Cobb et al., 1997, p. 221). Moreover, they explain how their analysis of the classroom activity is consistent with the claim that "tools are not merely amplifiers of human capabilities, but they lead to the reorganization and restructuring of activity" (Cobb et al., 1997, p. 222). In another example, Cobb and Bowers (1998, p. 111) explain how mediating artifacts are central to RME:

> From [our] viewpoint, the use of tools is viewed as integral to mathematical activity rather than an external aid to internal cognitive processes located in the head (cf. Hutchins, 1995; Meira, 1998; Pea, 1993). From this perspective, it therefore makes sense to speak of students reasoning with physical materials, pictures, diagrams, and computer graphics as well as with conventional written symbols. This nondualist focus on tool use is, in fact, central to the RME design theory that guides our development of instructional sequences.

To a large extent, the arguments made by Cobb and colleagues above are those that I have made in this paper. However, while I claim that these arguments necessitate a cultural perspective within RME, for Cobb et al., the connection between RME and cultural theories of learning remains only partial. There are two reasons for this: First, Cobb and colleagues find a disparity in the way that learning is said to occur in the two approaches, and second, they find a disparity in the way that individuals are characterized. Below I address each of these perceived disparities, and argue that there is, in fact, no disparity in either case.

<u>Disparity #1: How learning is said to occur</u>: Cobb and colleagues interpret cultural theories of learning as implying that learning is "a process of *transmitting* mathematical meaning from one generation to the next" (Cobb et al., 2008, p. 110, italics added). They contrast this with RME, for which learning is a process of "*emergence* of mathematical meaning in the classroom" (Cobb et al., 2008, p. 110, italics added). I certainly agree with Cobb et al. that RME theorizes

learning as a process of emergence of mathematical meaning. On the other hand, my reading of the literature suggests that contrary to Cobb et al., cultural theories of learning do not call for transmission pedagogy. In contrast, the notion that meaning emerges from activity is perfectly consistent with cultural theories of learning.

The association between cultural theories of learning and transmission pedagogy seems to stem from a somewhat common, but arguably narrow, reading of Vygotsky's writing on the social situation of development and internalization of cultural forms. While Vygotsky was interested in internalization, he rejected the notion that culture could simply be transmitted:

> It is impossible to exert a direct influence on, to produce changes in, another individual… reduc[ing] the process of education and instruction to a passive apprehension by the student of a teacher's lessons and outlines [is] the ultimate of psychological nonsense (Vygotsky, 1997a, pp. 47–48)

Furthermore, the notion that meaning is an emergent phenomenon that happens through activity is perfectly consistent with cultural theories of learning, as the quote below makes clear (see also, Radford, 2000, 2003).

> From the time of their earliest publications in the late 1920's and early 1930's, the Russian cultural historical psychologists emphasized the tripartite nature of human mental processes. They represented the basic structure of consciousness as the *emergent* process involving an active subject, an object, and the cultural medium (Cole & Levitin, 2000, p. 65, italics added).

RME, with its focus on emergent modeling, offers a particularly refined theory of how mathematical meaning emerges through activity. Such an emergent process is perfectly compatible with a cultural perspective.

Disparity #2: The role of the individual. Cobb and colleagues interpret distributed accounts of cognition as "dismiss[ing] the individual from theoretical consideration" (Cobb et al., 1997, p. 227), and they argue for a perspective that preserves the "active individual" (Cobb, Stephan, McClain, & Gravemeijer, 2001, p. 105). In order to preserve the active individual, Cobb and colleagues argue that one must reject cultural theories of learning, and must instead import ideas from psychological constructivism.

In my reading of the literature, I find that cultural perspectives have always included a place for active individuals (Cole & Wertsch, 1996). For example, Vygotsky (1997a, p. 48) explained that "the educational experience must be based on the student's individual activity, and the art of education should involve nothing more than guiding and monitoring this activity." Thus, as Cole and Wertsch (1996) make clear, the common notion that cultural perspectives reject the active individual is misguided. One needn't turn to psychological constructivism to preserve an active individual. However, the ontological nature of the "active individual" is different from a cultural perspective:

> The knowing and learning individual is both active and acted on. When constructivism assumes that this activity is always intellectual and individual it fails to grasp the affective, relational, and cultural dimensions of activity. (…) [a cultural perspective]

> envisions a practical process of construction where people shape the social world, and in doing so are themselves transformed (Packer & Goicoechea, 2000, pp. 234–235).

I will return to the ontological differences in the next section. To sum up my argument in this section, Cobb and colleagues largely agree with my analysis of the role of mediation within RME. However, whereas I have argued that a cultural perspective is necessary within RME, Cobb et al. came to a different conclusion, noting only a "possible point of contact" between RME and cultural theories of learning. They did so because of two perceived disparities between RME and cultural theories of learning: (1) a disparity in how learning is theorized to occur, and (2) a disparity in the role of the individual. In this section I have argued that these two disparities are ephemeral, and that a cultural perspective is fully compatible with RME. This is important, because a cultural perspective carries multiple implications for RME, which I summarize in the next section.

5. Implications for RME of adopting a cultural perspective

The main implication to RME of adopting a cultural perspective is a shift in perspective, away from a focus on individuals, and towards a focus on systems of persons and artifacts engaged in activity. Understanding the implications of this shift in perspective is a task for future work. However, below I sketch some possible implications for three of the principles of RME that I summarized in Section 1, and I introduce a new principle that I argue should be incorporated into RME. My exposition of these implications is brief. My goal here is not to settle anything, but rather to point to possible implications that should be studied in future research.

Implications to the activity principle

Freudenthal (1991) characterized mathematics as a “mental activity” (p. 2) and a “private activity” (p. 14). From a cultural perspective, such a view is much too narrow. As I have argued in this paper, mathematizing is a mediated activity, and therefore cannot be located solely within an individual:

> Because what we call mind works through artifacts, it cannot be unconditionally bounded by the head or even by the body but must be seen as distributed in the artifacts that are woven together and that weave together individual human actions (Cole, 1995, p. 110).

At the very least, mathematizing is a “mediated action” (Wertsch, 1994, 1998)—that is, an “individual(s)-operating-with-mediational-means” (Wertsch, Tulviste, & Hagstrom, 1993, p. 343). Often, however, individual actions with mediational means cannot be understood except as part of a larger constellation of interactive activity. For example, earlier I described how David marshaled talk, gesture, and projected inscription into a semiotic bundle that mediated the mapping of numeric values onto physical space. I further described how the gesture itself became a representational artifact, which Stacy incorporated into her action to mobilize David’s strategy into the triad. Each of actions—David’s use of the semiotic bundle to map numbers onto space, and the Stacy’s subsequent use of the gesture—is a mediated action, but neither can be understood except as *moment* of the larger group’s activity. In these cases, a more expansive notion of activity is required:

> [T]he mediated actions and interconnected sequences of actions (i.e. operations) that individuals carry out in the attainment of a goal. (…) [I]n the course of the activity, individuals relate not only to the world of objects (the subject-object plane) but also to other individuals (the subject-subject plane or plane of social interaction) and acquire, in the joint pursuit of the goal and in the social use of signs and tools, human experience (Radford et al., 2007, p. 512)

Such a definition seems to best capture the examples of mathematizing that I have given in this paper (the Weight Watcher and the students reinventing the algebraic ratio). Furthermore, this expansive definition seems to be most appropriate to educational contexts, where the limited notion of an individual operating with mediational means is likely insufficient to provide an adequate account of learning. As Dewey (1916, p. 19) explained, "things gain their meaning by being used in a shared experience or joint action."

For RME researchers, the implication is that joint mediated activity is an appropriate unit of analysis for studying mathematizing. Researchers should pay particular attention to the constituting role of artifacts in human activity and human mental functioning. For RME designers, the implication is to create and study "artifact-saturated environments" (Gutiérrez, 2011, p. 32) for mathematizing.

Implications to the reality principle

Freudenthal (1983, 1991) took a phenomenological approach to reality, stating:

> I prefer to apply the term “reality” to that which at a certain stage common sense experiences as real. (…) [Reality] is not bound to the space-time world. It includes mental objects and mental activities (Freudenthal, 1991, p. 17).

Thus, “in Freudenthal’s view, ‘common sense’ and ‘reality’ were construed from the viewpoint of the actor” (Gravemeijer & Terwel, 2000, p. 783). This perspective is problematic because it is easy to follow Freudenthal’s definition of an actor-construed reality all the way to solipsism. This, in turn, begs the question of why one should learn mathematics, if it is only to create an individual reality. Freudenthal’s (1968) answer was that mathematics is *useful*, “for the understanding and the technological control not only of the physical world but also of the social structure.” Of course, for mathematics to be useful to understand and control the physical and social world, it must have an existence beyond the individual.

One can understand the bind that Freudenthal was in. On the one hand, he was committed to the notion that mathematics could be experienced as real. On the other hand, he only considered two classes of objects: material and mental, and neither seem to capture the particular ontology of a useful mathematics, which I described in Section 2 as “durable and shared, existing across time and outside of any single individual.” In essence, Freudenthal’s bind is a version of the same dilemma that I introduced in Section 2 when discussing the nature of mathematical productions. The dilemma comes from dualist notions of persons and world, a position which states that “mathematical reality must lie either within us, or outside us” (White, 1947, p. 291).

Just as a cultural perspective resolved the dilemma in Section 2, so to does a cultural perspective resolve Freudenthal’s bind. From a cultural perspective, mathematics does have an objective existence: it exists objectively as a cultural artifact (Hersh, 1994; Radford, 2007; Roth

& Radford, 2011; White, 1947). It is a special kind of artifact, which Wartofsky (1979) calls a "tertiary artifact." Tertiary artifacts are those which have been "abstracted from their direct representational function" (p. 209) such that they:

> come to constitute a relatively autonomous 'world', in which the rules, conventions and outcomes no longer appear directly practical, or which, indeed, seem to constitute an arena of non-practical, or 'free' play or game activity (…) derived from and related to a given historical mode of perception, [but] no longer bound to it (pp. 208-109).

Hence a cultural perspective enriches the reality principle by explaining how mathematics can constitute a world that is more than just "experienced as real," but which is *actually* real, for a given time and within a given culture. As such it allows us avoid the solipsism inherent in Freudenthal's phenomenological account of mathematical reality. In addition, Wartofsky's notion of a tertiary artifact expands the usefulness of mathematics, beyond "the understanding and the control" of the physical and social world. Mathematics, like all tertiary artifacts, is a vehicle of social change:

> Once the visual picture can be 'lived in', perceptually, it can also come to color and change our perception of the 'actual' world, as envisioning possibilities in it not presently recognized.(…) The upshot [is] that the constructions of alternative imaginative perceptual modes, freed from the direct representation of ongoing forms of action, and relatively autonomous in this sense, feeds back into actual praxis, as a representation of possibilities which go beyond present actualities (Wartofsky, 1979, p. 209)

Because mathematics is means of social control and a vehicle of social change, learning mathematics is political. The implication for researchers and designers is to create and study opportunities for students to develop sociocritical literacies (Gutiérrez, Hunter, & Arzubiaga, 2009) and critical agency (Gutstein, 2006) as students *live* mathematics as a liberatory experience (Radford, 2012).

Implications to the interaction principle

A particularly unfortunate consequence of the individualistic focus in RME and the corresponding commodification of knowledge is an impoverishment in the interaction principle. As I described in Section 1, the interaction principle describes interaction as largely *transactional*, existing solely to enrich the individuals who engage in it. For example, Treffers (1987, p. 249) introduced the interaction principle by stating: “This means that the pupils are also confronted with the constructions and productions of their fellows, which can stimulate them.”

Cobb and colleagues (Cobb & Yackel, 1996; Yackel & Cobb, 1996) greatly enriched the interaction principle when they introduced the notion of a normative *classroom mathematical practice* that becomes “taken as shared” by participants. Such normative practices exist at a social level, “above” any individual student. Through interaction, participants do more than transact, they also contribute to a “taken as shared” world of normative practices. Cobb and colleagues noted a reflexive relationship between individual and normative ways of understanding. As such, they declined to give primacy to either. Each informs the other, however individual understandings remain private and unknowable by others, and thus social understandings are merely “taken as shared.”

A cultural perspective on RME builds on this important work and enriches the interaction principle even more. From a cultural perspective, interaction is the mechanism by which consciousness is created and objectively real mathematical objects come to have a shared meaning—actually shared, not just “taken as shared.” Interaction has this power because it is, by its very definition, a shared event. In interaction, “we produce action methodically to be recognized for what it is, and we recognize action because it is produced methodically in this way” (Heritage & Clayman, 2010, p. 10). In other words, interaction requires attunement to the Other. Interaction is a symmetric activity which participants, through successive production of—and reaction to—turns that are Other-oriented, create shared meaning. Here is where mathematical consciousness emerges. Consciousness doesn’t pre-exist, or stand in reflexive relation to the social, but rather it is *produced* in the shared social world: “consciousness constitutes, according to the etymology of the word, ‘knowing’ (Lat. sciēre) ‘together’ (Lat. Con-)” (Roth & Radford, 2011, p. 141). This is how mathematics can have an objective meaning. Participants work together in interaction to make culturally objective mathematical knowledge emerge in their shared consciousness (Roth & Radford, 2010, 2011).

In interaction, then, individuals are connected to the cultural world. Interaction is both the route by which the mathematical world becomes a place that can be “lived in” and the route by which this world “feeds back into actual praxis, as a representation of possibilities which go beyond present actualities” (Wartofsky, 1979, p. 209). Through interaction, people take their place as a “presence in the world” (Friere, 2004, p. 74; cf., Radford, 2012):

> That is to say, to become individuals who are more than *in* the world, individuals who relate to each other, intervene, transform, dream, apprehend, and hope. Becoming a

> presence in the world is not a natural process; it occurs against the background of history and culture. (…) And presence in the world is not about fleeing from cultural forms of thinking because they are not ours, because others have formed them before us. On the contrary, presence in the world requires the critical encounter with, and immersion in, those always evolving cultural-historical forms of thinking (Radford, 2012, p. 110).

For designers, the implication is to design interactive experiences that are more than simply opportunities for students to share and confront mathematical productions. More important is that students labor *together* such that they come to understand themselves and others as cultural, historical, and political persons. Such an understanding "consists not only of the mathematics ideas that students express in speech and deeds. Understanding is certainly this, but it is much more too. It is the understanding of *another presence*, and as such goes beyond the cognitive realm" (Radford, 2012, p. 111).

A new principle: The producer principle

As this paper has made clear, from a cultural perspective learning is much more than an epistemic activity. Ontology is implicated as well. Artifacts *act back* on people, interaction *produces* a shared consciousness, and through interactive activity, students *become* a presence in the common world. The upshot is that people are shaped and produced as particular kinds of people as they engage in joint mediated activity, even as people produce and shape the activity.

From this standpoint, identity is multiply-implicated: who students *are*[4], and who students are *becoming* are primary concerns. Students come to school with particular repertoires

[4] I realize that the phrase "who student *are*" conveys a static notion of identity. In using this term, I don't mean to convey such a notion, exactly. On the one hand, people have relatively durable subjectivities that have been

of practice (Gutiérrez & Rogoff, 2003), and having been produced in particular ways by their previous experiences. Up to now, RME has incorporated students' histories via Freudenthal's called for designers focus on creating contexts that are "experientially real" (Gravemeijer & Doorman, 1999, p. 111) for students, but I suggest that we can and should go further to design experiences that are *meaningful*, not just in a classroom but in students' lives. The implication is that designers should understand both the social realities of their students as well as the history and valued practices of the local community (Martin, 2007), and incorporate students' repertoires and histories meaningfully into classroom activities (Esmonde & Caswell, 2010).

We also need to consider *teloi* (endpoints) beyond epistemic objectives. This requires a new way of thinking about mathematics within RME, which has, up to this point, theorized mathematics as an activity and as a product. In addition, instructors, designers, and researchers in RME ought to also consider mathematics as a *producer*, and learning and doing mathematics as involving the production of persons (Bishop, 1991; Packer & Goicoechea, 2000; Radford, 2008b) The questions for instructors, designers, and researchers are, what sort of person is consistent with the vision that guides RME? What sort of people do we want our students to become (Packer, 2001)? I consider this question to be the most pressing question for future research in RME.

6. Conclusion

I opened this paper with a quote from Keono Gravemeijer (1999, p. 159), in which he states that, within RME-based research, "each time the research question is: What would mathematics education, which fulfills the initial points of departure, look like[?]" This paper has been my response to that question.

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produced across time (Bourdieu, 1972; Halperin, 2012). On the other hand, people actively create and recreate

Within RME, mathematics-as-product has been theorized as a collection of "thought objects" and learning has been taken up as the acquisition of "private knowledge." I have demonstrated that such a position is in tension with the first principles of RME, namely the activity principle and the reinvention principle. I resolved this tension by showing that mathematical objects are cultural objects, and by showing how a cultural perspective, which accounts for the constituent role of artifacts in human activity, is necessary if we are to "fulfill the initial points of departure" of RME. I exemplified this position with an empirical example of the quintessential form of mathematizing in RME: reinvention.

A cultural perspective enriches RME's instructional principles, with implications to the activity principle, the reality principle, and the interaction principle. I have also suggested that a new principle ought to be incorporated into RME: the producer principle, which states that learning mathematics involves the production of persons. Exploring and refining these implications is the next frontier for research in Realistic Mathematics Education.

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themselves—and are being created and recreated—as they participate in activity (Dreier, 1999, 2009).

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