Reinventing Fractions and Division as they Are Used in Algebra: The Power of Preformal Productions

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Abstract

In this paper, we explore algebra students' mathematical realities around fractions and division, and the ways in which students reinvented mathematical productions involving fractions and division. We find that algebra students' initial realities do not include the fraction-as-quotient sub-construct. This can be problematic because in algebra, quotients are almost always represented as fractions. In a design experiment, students progressively reinvented the fraction-as-quotient sub-construct. Analyzing this experiment, we find that a particular type of mathematical production, which we call preformal productions, played two meditational roles: (1) they mediated mathematical activity, and (2) they mediated the reinvention of more formal mathematical productions. We suggest that preformal productions may emerge even when they are not designed for, and we show how preformal productions embody historic classroom activity and social interaction.

Keywords: Realistic Mathematics Education, RME, Fractions, Division, Algebra, Preformal Productions

Reinventing Fractions and Division as they Are Used in Algebra: The Power of Preformal Productions

The study reported in this paper was motivated by our observations—as educators and researchers in Algebra I classrooms—that there is often a mismatch between students' prior experience with fractions and the way that teachers use fractions in introductory algebra. How are fractions used in algebra? Consider Equations 1and 2 below, where, for each, the task is to solve for the unknown variable:

$$4x - 8 = 0 \tag{1}$$
$$9y - 7 = 0 \tag{2}$$

Superficially, there seems to be little difference between the two equations. However, there is one key difference: Equation 1 has an integral solution, while Equation 2 does not. In our experience, this difference makes the two tasks very different for students in introductory algebra. The difficulty occurs at the "division step" in the traditional algorithm for solving equations (for example, in Equation 2, the division step would come after adding 7 to both sides of the equation, and it would involve dividing 7 by 9 in order to find *y*). Students' solution processes up to the division step are similar for both equations. However, we see many more mistakes in the division step for equations without integral solutions, like Equation 2. These mistakes take many forms. Some students will state that the problem is not solvable, because, to take Equation 2 as an example, "you can't divide 7 by 9." Other students do the division "backwards" (e.g., they solve Equation 2 by dividing 9 by 7).

Furthermore, in our experience nearly all introductory algebra students who attempt the division use the division symbol (÷) to represent the division operation, and use the long division algorithm to express their final solution in decimal notation. Few students use the fraction bar to represent the division operation or use fractions to represent their solutions. This last point may seem trivial: why should it matter if a correct answer is expressed in decimal notation or fraction notation? However, it is important because it suggests that students do not seem to recognize that a fraction can be used to represent, simultaneously, a division problem and the numerical result of the division problem (this is often referred to as the "quotient" sub-construct of rational number, Kieren, 1980; see the review of the literature on fractions as quotients below).

If this is true, it is especially concerning because the fraction bar will serve as a division symbol, and hence fractions will serve as quotients, for the rest of a student's mathematical life (Rotman, 1991). Furthermore, fraction vocabulary will come to be synonymous with division vocabulary. For example, the slope of a linear function can be conceptualized as the change in the dependent variable per unit change in the independent variable. Such unit rates are calculated by dividing, but they are represented by fractions (e.g., $\frac{\Delta y}{\Delta x}$), and they are verbalized using language that only make sense in terms of fractional representations (e.g., "delta *y* over delta *x*" or "rise over run," where the word "over" designates a position in a fraction, but mathematically can be interpreted as a division operator).

It is therefore vital that students have experiences in which they come to understand that fractions can represent quotients, and that the fraction bar can be interpreted as a division operator. Our observations (discussed above) suggest that many students who enter our introductory algebra course have not had such experiences. But is this really the case? If so, how can it be addressed in an algebra classroom? To explore these questions, we engaged in a design experiment (Cobb, Confrey, Lehrer, & Schauble, 2003; Steffe & Thompson, 2000), the results of which are summarized in this paper.

To begin, we will discuss our conceptual framework and then summarize the relevant literature on fractions-as-quotients and on design experiments on fractions. In doing so, we situate our study in the vast literature on how students learn fractions. We will then describe the design of our experiment and our specific research questions. We follow this with a description of the learning activities that emerged in the experiment. Finally, we discuss the pertinent results and highlight our contributions to theory and practice.

Conceptual Framework and Literature Review

Realistic Mathematics Education

Our design experiment was guided by the principles of Realistic Mathematics Education, (RME; Freudenthal, 1973, 1991), which is based on Hans Freudenthal's belief that mathematics is not a ready-made structure, but rather the *human activity of structuring the world mathematically* (which Freudenthal called "mathematizing").

This belief led Freudenthal to conclude that mathematics education should not be concerned with the transmission (or even the discovery) of formal mathematics, but rather with engaging students in the activity of mathematizing. He recognized that in order to be meaningful, such activities had to be rooted in the student's reality. However, this did not mean that Freudenthal rejected formal and abstract mathematics. On the contrary, he observed that mathematicians use and discuss abstract (and in some sense, imaginary; Núñez, 2009) mathematical productions as if they were real objects. Indeed, for the mathematician, these imaginary productions are real objects, and formal mathematics is reality.

Mathematics education, then, should involve engaging students in activity such that mathematical productions become real to students. Broadly, mathematical productions can be categorized at three levels of formalization (Webb, Boswinkel, & Dekker, 2008). Students begin by mathematizing contextual situations. As students engage with these problems, they draw pictures and create models. These are models *of* learning (Gravemeijer, 1999), and are specific to the problem at hand. These are called *informal* models. Through further problem solving, these informal models can themselves be mathematized, leading to more general models. These are models *for* learning (Gravemeijer, 1999), and they can be applied to problems beyond the problem at hand. These are called *preformal* models. Further problems encourage students to formalize preformal models into formal mathematics. *Formal* productions are stripped of all contextual clues, making them potentially very general, but also very abstract.

This process of progressive formalization (van Reeuwijk, 2001) has been theorized as a process of *emergent modeling* (Gravemeijer, 1999), in which models are reinvented through a *chain of signification* (Gravemeijer, 1999; Whitson, 1997). At first, informal models signify a particular context or situation. This forms a "sign" that consists of a *signifier* (the informal model) and a *signified* (the problem context). In subsequent activity, this sign is itself signified by more general models. The process continues such that models at one level signify those signs that came before, and the new sign is signified by those that come after. We draw on this notion to define *learning* as the reinvention of mathematical productions, and conceptualize the process of learning formal mathematics as one of emergent modeling via a chain of signification. However, in this paper, we do not limit ourselves to models when we discuss the productions that form the chain of signification. Instead we demonstrate the value of a broader view, and we include all manner of productions, including models, tools, and strategies.

Chains of signification are created as students engage in mathematical activity. Problem situations play a key role: they set the stage for students' initial informal productions, and they drive the creation of more formal productions. The focus on thoughtfully designed problem situations in RME is reminiscent of the Theory of Didactical Situations (TDS; G. Brousseau, 1997). In both RME and TDS, students are presented with problem situations that are "begging to be organized"

(Gravemeijer & Terwel, 2000), and which the students can take ownership of. In TDS, these well-designed problem situations are called *adidactical situations*. The term "situation" has a broader meaning in TDS, encompassing the task, the students, the teacher, and the *milieu* in which teaching and learning take place. There is no question that the elements of this broader situation play important mediating roles in learning, and more recent conceptions of RME (Cobb, Zhao, & Visnovska, 2008) have attempted to incorporate this broader perspective. In our discussion section, we consider the mediating role of the teacher. However, for most of this paper we focus on the problem situations, the students, and the mathematical productions that emerged as students interacted with the problem situations.

Literature review: Rational numbers as quotients

Kieren (1980) identified five sub-constructs for rational numbers: (1) Part/whole, (2) Ratio, (3) Quotient, (4) Measurement, and (5) Operator. While these sub-constructs are related, the contexts in which they arise are very different. For example, consider how the fraction $\frac{3}{4}$ might be interpreted under the part/whole and quotient sub-constructs: (Lamon, 2005):

> **Part/whole:** I have one pizza, cut into four equal pieces. If I eat three of those pieces, I have eaten $\frac{3}{4}$ of the pizza.
>
> **Quotient:** If three pizzas are shared equally amongst four people, each person receives $\frac{3}{4}$ of a pizza. Hence, $3 \div 4 = \frac{3}{4}$

Before discussing the literature on rational numbers as quotients, we first have to address an issue of terminology regarding the terms "fraction" and "rational number." Following Lamon (2007), we define *fraction* as a form of notation that expresses a multiplicative relationship numerically as $\frac{a}{b}$. We define *rational numbers* as the set of numbers that can be expressed as the quotient of two integers. Rational numbers may be written as fractions, but they can be expressed in other ways as well (e.g., $\frac{1}{2}$ and 0.5). At the same time, two different fractions (e.g., $\frac{1}{2}$ and $\frac{2}{4}$) may represent the same rational number. Finally, it is possible to express non-rational numbers as fractions (e.g., $\frac{\pi}{2}$). In this paper, we are often interested in the fraction representation. When we mean to invoke the $\frac{a}{b}$ notation, we use the term "fraction."

The part-whole construct is the canonical sub-construct for most students and schooled adults (Lamon, 2007). Students are more successful when solving problems that involve the part-whole sub-construct than they are when solving problems involving the other sub-constructs, and mastery of the part-whole sub-construct is only very weakly correlated with mastery of the quotient sub-construct (Charalambous & Pitta-Pantazi, 2006; Clarke, Roche, & Mitchell, 2007). Furthermore, evidence suggests that overemphasis of the part-whole sub-construct can actually be detrimental to the development of the other sub-constructs (Pitkethly & Hunting, 1996).

These results suggest that the quotient sub-construct is qualitatively different than the part-whole sub-construct, and that students would benefit from more experience with problem situations—such as fair-sharing multiple whole amongst multiple people—that lead to an understanding of fractions as quotients. By mathematizing such contexts, students can come to see that if *a* items are shared amongst *b* people, each person receives $\frac{a}{b}$ items. Further abstraction and formalization leads to the general statement that $a \div b = \frac{a}{b}$ (Confrey, 2012; Empson, 1999; Fosnot & Dolk, 2002; Streefland, 1993).

It is worth noting, however, that while the idea of sharing is natural for students, the process of finding the amount that each sharer receives is not. For one, sharing is *partitive division*, which is generally more difficult for students to construct than *quotative division* (Fosnot & Dolk, 2001). This is because partitive division involves problems in which the dividend and the divisor do not have the same units and hence, repeated subtraction is not a valid strategy.

Consider the task of sharing 21 beads amongst three strings. As documented in Fosnot & Dolk (2001), students first approach this problem through trial-and-error by guessing at the number of beads on each string and then checking to see if their guess results in all 21 beads being distributed equally. Eventually, students invent a "dealing-out" strategy, in which one bead is "dealt" to each strand, followed by a second bead to each strand, and so on until the beads are exhausted (cf. Wilson, Myers, Edgington, & Confrey, 2012). In the language of progressive formalization, this "dealing out" strategy has the potential to be a preformal strategy because although it does not take advantage of formal mathematics, it can be made general enough to be applied to any fair sharing problem (and can subsequently be made general enough to apply to any partitive division situation).

Applying the same strategy to situations in which each sharer receives a non-whole amount is yet more complicated, because the student has to (a) partition the wholes into a number of pieces (unit fractions) that can be distributed equally, (b) reunitize each unit fraction in order to operate on it independently, (c) distribute the unit fractions equally amongst the sharers (possibly using the "dealing-out" strategy), and finally (d) iterate the distributed unit fractions to create a non-unit fraction, which describes the amount that each sharer receives (Lamon, 1996; Olive & Steffe, 2001; Olive, 1999). We call this the "partition-distribute-iterate" strategy.

Students construct a number of strategies when they solve fair-sharing problems (Charles & Nason, 2000; Empson, Junk, Dominguez, & Turner, 2006; Empson, 2002). For example, Empson et al. (2006) classified strategies according to how the students coordinated the number of items to be shared with the number of sharers. *Precoordinating strategies* are distinguished by the lack of coordination between the number of items to be shared and the number of sharers. Strategies in this category include giving unequal amounts to the sharers, or giving equal amounts without exhausting the quantity to be shared. *Coordinating strategies* are distinguished by partitions that create a number of parts that is either equal to or a multiple of the number of sharers. For example, each item to be shared may be partitioned into the number of sharers (e.g., sharing fours pizzas among six people by partitioning each pizza into sixths). Using the language of progressive formalization, we consider precoordinating strategies and to be *informal* strategies because they are not successful across a wide variety of problems. We consider coordinating strategies to be *preformal* strategies because they are general strategies that can be used successfully across problems.

In addition to strategies, researchers have explored various models that students use as the engage in equipartitioning activities. While the circle model seems to be the canonical model for fractions, many students find it difficult to partition circles (Ball, 1993; Confrey, 2012), and many researchers have found that a "bar model" (i.e., a rectangle) is easier for students to work with (Connell & Peck, 1993; Keijzer & Terwel, 2001; Middleton, van den Heuvel-Panhuizen, & Shew, 1998; Moss & Case, 2011).

Literature review: Teaching experiments on rational numbers

Student learning of rational numbers has received an immense amount of research scrutiny (e.g., Pitkethly & Hunting, 1996). The research related to the quotient sub-construct is summarized above. In this section we broaden the lens, and summarize four seminal studies involving teaching experiments from the US, France, Russia, and the Netherlands. We then situate our study in this literature.

In the US, the Rational Number Project (RNP) has been conducting teaching experiments on rational numbers for over 30 years (Behr & Post, 1992; Cramer, Post, & delMas, 2002). Researchers on the project have studied all aspects of rational number, and have distilled their findings into a two-year curriculum for rational numbers (Cramer, Behr, Post, & Lesh, 2009; Cramer, Wyberg, & Leavitt, 2009). Through their work, researchers on the RNP have convincingly demonstrated how "understanding is reflected in the ability to represent mathematical ideas in multiple ways, plus the ability to make connections among the different embodiments" (Cramer, 2003, p. 450).

In France, G. Brousseau and N. Brousseau (N. Brousseau & G. Brousseau 1987; English translations in G. Brousseau, N. Brousseau, & Warfield, 2004, 2007, 2008, 2009) designed a curriculum for rational number as a proof-of-concept for G. Brousseau's (1997) theory of didactical situations. Guided by theory, the curriculum includes 65 imaginative and well thought-out lessons which have been taught and refined in teaching experiments for over 15 years. Through engaging in meticulously designed situations, students "invent, understand and become fluent with all the aspects of [rational numbers]" (G. Brousseau et al., 2007, p. 281). Students begin by inventing rational numbers as measures. By the end of the 65 lessons, students have invented all of the sub-constructs of rational number, fraction and decimal notation, formal operations on rational numbers and decimals, and the topology of rational numbers.

A somewhat similar sequence comes from Russia, in the work of Davydov and colleagues (Davydov, 1990; see also Schmittau & Morris, 2004). Davydov describes a curriculum for rational number that is also organized around measurement and which also results in students' understanding the topology of the rational numbers. Davydov's curriculum, however, has vastly different theoretical underpinnings. It is informed by activity theory (Schmittau, 2003), and takes an algebraic approach to number in which students learn rational numbers by ascending from the abstract to the concrete (Falmagne, 1995).

In the Netherlands, Streefland (1991, 1993) conducted teaching experiments over a 10-year period on a series of lessons for rational number designed using RME principles. In these lessons students engage in two types of activities: fair sharing and splitting the group of sharers. The activities were designed such that multiple sub-constructs of rational number were intertwined from the beginning, and as students engaged in the activities, they reinvented all of the sub-constructs of rational number.

Although each of the above research projects took place in different countries and had different foci and theoretical underpinnings, there are some common themes. First each research project explored students' initial exposure to rational numbers. Second, the projects were quite comprehensive, exploring how students learned multiple sub-constructs of rational number, as well as formal operations on rational number, and—in some cases—the topology of rational number. The study that we describe in this paper differs from the above studies in both respects. First, our study involves students who have vast prior experience with rational numbers. Second, our intention is not to understand how students come to a comprehensive understanding of rational numbers, but instead to explore how students come to understand fractions as they are used in algebra.

Research questions

Our study was guided by the following two research questions:

RQ 1. How do our students solve partitive division problems with integral and non-integral results?

RQ 2. How do our students reinvent the fraction-as-quotient sub-construct?

Notice that we are interested in *how* our students solve problems and reinvent the fraction-as-quotient sub-construct. As detailed in our conceptual framework, we believe that reinvention happens during mathematical activity, and thus we designed a sequence of activities to encourage students to reinvent the fraction-as-quotient sub-construct. In what follows, we present the sequence of activities along with our design rationale. We do so in order to communicate the conditions under which our study took place. We do not intend that the activities should serve as a "model" curriculum. Rather, the descriptions set the stage for our analysis of how students reinvented the fraction-as-quotient sub-construct.

Materials and Methods of Analysis

A two-person research team (the authors of this paper) conducted the design experiment: a teacher-researcher (FAP, the first author), and an observer (MM, the second author). The learning activities took place in a public high-school in a suburban area of the United States (US). In the US, all students are required to attend high school, and with rare exceptions students are not sorted into specific schools. The vast majority of students attend so-called "comprehensive high schools," which is the type of school in which our experiment took place. FAP was the teacher of the class in which the study took place (Cobb, 2000), and the entire class participated in the learning activities. The school served a predominantly white (approximately 60%) and Latino (approximately 30%) population. We do not have access to student-specific demographic data.

The course itself was a support class for ninth-grade students that were concurrently enrolled in Algebra I. Algebra I is a common course in US high schools, and is generally taken in ninth-grade (the first year of high school in the US; students in ninth-grade are about 14 years old). In the study high school, Algebra I is a required course for all students. At the time of the study, the school's curriculum for Algebra I included solving single-variable linear equations and systems of two linear equations, and in-depth study of linear and quadratic functions.

Students were assigned to the support course based on the recommendation of their Algebra I teachers, as well as on the basis of their scores on the previous year's state-level standardized test. With respect to teacher recommendation, Algebra I teachers were not given official criteria to use when recommending students. In general, teachers recommended students for the course based on the teacher's subjective opinion that the student would benefit from having more time to explore Algebra I concepts in a small-class setting. This recommendation was cross-referenced with the student's score on the state standardized test. The state categorized students into four proficiency levels based on their scores, and only students who were categorized in the lowest two proficiency levels were selected for the support course. In total, there were 12 students in the course.

Thus the students in our study are not representative of the Algebra I students at the school, or of some larger population of Algebra I students. However, our intention is not to make a claim about the knowledge-level of some generalized population of Algebra I students, or about the effectiveness of a particular curriculum. Rather, as discussed above, we want to explore how students who are enrolled in high school Algebra I reinvent the fraction-as-quotient sub-construct.

In the course, students engaged primarily in activities that were correlated with the concurrent Algebra I topics, however, occasionally students also engaged in activities that were designed to increase familiarity with signed integers and rational numbers (for a description of such an approach, see Burris & Welner, 2005). FAP designed the entire course using RME principles. Students were accustomed to engaging in mathematical activity prior to being taught specific procedures, and they were accustomed to sharing strategies through "math congress" (Fosnot & Dolk, 2001, 2002), in which a carefully sequenced subset of students presented their solution strategies to the class.

Prior to beginning the experiment, we developed a hypothetical learning trajectory (elaborated below). The experiment itself consisted of seven learning activities, each of which took a full (55-minute) class session. In order to keep the primary focus of the support course on the concurrent Algebra I topics, we distributed these learning activities throughout a two-month period, giving approximately one learning activity per week. After each learning activity, we met as a research team to examine student work, update our models of students' mathematical realities, discuss our impressions of how the learning activity influenced those realities, and design subsequent learning activities. In this, our approach was cyclical: "What is invented behind the desk is immediately put into practice; what happens in the classroom is consequently analyzed, and the result of this analysis is used to continue the developmental work" (Gravemeijer, 1994, p. 449).

Data Sources

During the experiment, we collected student work and MM recorded fieldnotes. When he felt it was appropriate, MM recorded classroom discourse in the fieldnotes. In addition, we kept a record of our meetings, and saved all analytical memos that we sent each other during the experiment. As discussed above, the initial analysis happened concurrently with the design experiment. We further analyzed the collected and created artifacts after the conclusion of the experiment.

Hypothetical Learning Trajectory

In a design experiment, the Hypothetical Learning Trajectory (HLT) is created *a priori*, and represents "a prediction as to the path by which learning might proceed" (Simon, 1995, p 135). Because we define learning as the reinvention of progressively more-formal mathematical productions, our HLT describes a path of progressive formalization, as follows:

Stage 1: Students use informal models and strategies to solve problems involving fair sharing situations where multiple items are shared amongst multiple sharers.

Stage 2: Through social interaction (including math congress) and further experience with progressively more abstract fair-sharing situations, students develop the bar model and the "partition-distribute-iterate" strategy.

Stage 3: Students formalize the fractions-as-quotients sub-construct.

The HLT is only a prediction about what might happen, and the actual path of learning emerges in the design experiment itself as the research team interacts with student learning. In the next two sections we describe the path of learning that emerged in our design experiment.

Exploring Research Question 1, and Building an Instructional Starting Point

Our first research question was focused on how our students initially solved partitive division problems. As such, exploring this question provides us with the "instructional starting point" (Gravemeijer & Cobb, 2006) for the learning activities to follow. Based on the literature described above, we chose to begin with a problem designed to elicit student strategies and models in a fair-sharing context. This lead to the design of Learning Activity 1.

Learning Activity 1: The sub-sandwich problem

Motivation and design

In order to explore the ways that students solve partitive division problems, we began our experiment with a problem from a fair-sharing-based curriculum on rational number (Fosnot, 2007), henceforth called the "sub sandwich problem" (see Figure 1).

> A class traveled on a field trip in four separate cars. The school provided a lunch of submarine sandwiches for each group. When they stopped for lunch, the subs were cut and shared as follows:
>
> - *The first group had 4 people and shared 3 subs equally.*
> - *The second group had 5 people and shared 4 subs equally.*
> - *The third group had 8 people and shared 7 subs equally.*
> - *The last group had 5 people and shared 3 subs equally.*
>
> When they returned from the field trip, the children began to argue that the distribution of sandwiches had not been fair, that some children got more to eat than the others. Were they right? Or did everyone get the same amount?

Fig. 1 The sub-sandwich problem from Fosnot (2007)

From an RME standpoint this is a very good introductory problem for the following reasons:

1. The context can be made real to students, which we did by having a class discussion about sub-sandwiches before giving the task. Furthermore, the question of fairness is motivating (Paley, 1986).

2. Informal models *of* the situation (pictures of sub-sandwiches) are similar in shape to the bar model, and sub-sandwiches can be physically cut and distributed in a way that is similar to the partition-distribute-iterate strategy. Thus, the informal models and strategies that we predicted that students would use in the sub-sandwich problem can be mathematized to create the preformal bar model and partition-distribute-iterate strategy.

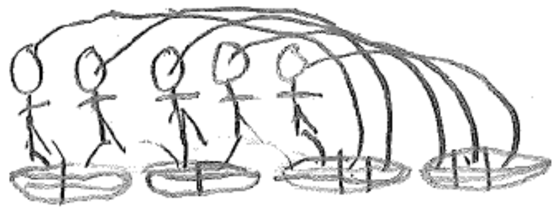
Because we used this activity to build our initial models of students' mathematical realities, students worked on the problem individually. We asked questions to probe for students thinking, but we did not offer help or suggestions.

Analysis

In order to build our initial model of students' mathematical realities, we coded the student work for level of formalization. We did this as a team, and came to a consensus for each student. Students used productions at all levels of formalization. For space purposes, we will limit our discussion below to the models and strategies that students used to find the quantity allotted to each person in Group 2 (four sandwiches shared amongst five people) as these productions are illustrative of the productions that students used for the other parts of the problem.

Informal productions: Two examples of students who used informal productions are shown in Figure 2. Figure 2a shows an example of a student who used informal models. These were models *of* the situation: pictures of sandwiches and people, with lines drawn to show how the sandwiches were distributed to the people. Both examples demonstrate informal precoordinating strategies, in which the partitions are based on benchmark fractions. None of the students who used informal strategies shared the sandwiches equally.

a. Penny:



b. Julia:

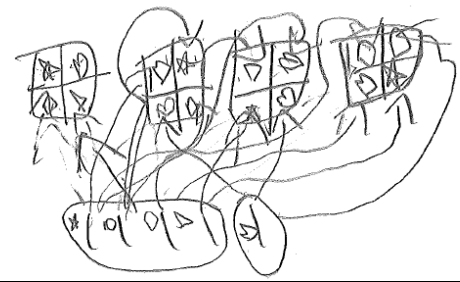


Fig. 2 Two examples of students who used informal models and strategies

Preformal productions: Preformal models were models *for* the situation (e.g., bar model). Preformal strategies were *coordinating strategies*. Figure 3 shows an example of a student who used a pre-formal bar model and a coordinating strategy.

Fig. 3 An example of a student (Theo) who used a preformal model and strategy

Formal productions: Students who used formal productions expressed quantities in symbolic fraction notation without any accompanying pictures or other work. When prompted, none of these students could draw pictures to connect their formal notation to the problem context, nor could they offer a mathematical justification as to why their solution was correct. For example, Justin and Joshua were sitting next to each other. As shown in Figure 4, they each wrote fractions that were reciprocals of each other. FAP asked Joshua and Justin to justify to try to convince each other why their solutions were correct. In response, Joshua drew a single bar model, cut into fifths, with four fifths shaded (demonstrating the part-whole sub-construct of fraction). However, he did not explain how this model was related to the problem situation of sharing four sub-sandwiches to five people. Justin justified his solution by pointing to the numerals (5 and 4) in the problem. When FAP asked, neither Justin nor Joshua was convinced by the other to change their solution.

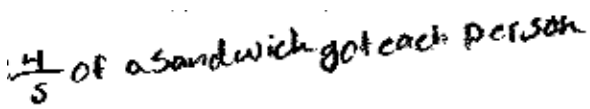
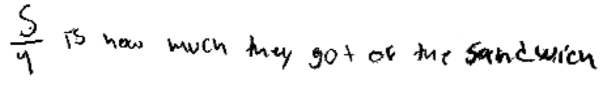
a. Joshua: Correct answer	b. Justin: Incorrect answer
	

Fig. 4 Two examples of students that used formal fraction symbols to represent the quantity

Analysis: When we created our HLT, we hypothesized that at the beginning of the experiment students would use informal models and strategies in fair-sharing situations. Through this learning activity, we learned that our students marshaled productions at all levels of formalization when solving fair-sharing problems. However, only the students who used preformal productions solved the problem correctly and justified their reasoning in the context of the problem. This suggested to us that preformal productions are a vital part of a student's mathematical reality.

We also learned that students might not associate fair sharing with the division operation, as only three students mentioned division in their written work or verbal descriptions. This suggested that students might not recognize partitive division situations as opportunities to use the division operation. We explored this with our next learning activity.

Learning activity 2: What operation?

Motivation and design

In order to explore the operations that students associate with partitive division situations, we designed an online problem-solving environment in which students were presented with a series of problems and an on-screen calculator. Students could use the calculator to perform any arithmetic operation, and they were provided a space to enter their final answer. They were also given the option of stating that a problem could not be solved. We used a screen-capture utility to record the students' activities in this problem-solving environment. This way, we were able to capture the operations that students associated with each problem.

Students worked on the problems individually. During the activity, we monitored students to ensure that there we no technical problems, but we did not provide problem-solving assistance. As in Learning Activity 1, we made this decision because we used this activity primarily to help build our models of students' initial mathematical realities.

After the activity we saved the screen capture videos, and later transcribed the students' keystrokes. We each watched each video at least twice: once independently and once as a research group. Our primary focus during this analysis was on whether, when, and how students used the division operation. We therefore coded each (student, problem) instance in terms of whether the division operation was used correctly, and if so, whether the division operation was the first and/or only operation used. Occasionally, students entered a final answer without using the calculator; we coded this as "strategy unknown."

Analysis

Nearly all of the partitive division problems were solved using the correct division operation. However, the correct division was the *only* operation just 70% of the time (this and other percentages presented in this paper are presented for summary purposes only, and are rounded to the nearest 5%). In other words, although many students ultimately settled on the answer provided by the correct division operation, many times students tried more than one operation.

We investigated these "multiple operation" instances further, and found that two students, Justin and Robert, consistently divided "both ways," and then chose one of the answers. For example, Protocol A shows how Robert solved a problem by dividing both ways.

Calculator: 3/57 = 0.05263…
Calculator: CLEAR
Calculator: 57/3=19
Types: 19

Protocol A Robert's "divide both ways" strategy for a problem that involved finding the weight of one chicken given that three chickens weigh 57 pounds

This was not the first time that we had observed such reciprocal division. For example, recall Justin's work on the sub sandwich problem (Figure 4b). Justin used formal notation to express the quantity of sub sandwiches that each person received, but his fraction ($\frac{5}{4}$) was the reciprocal of the correct fraction ($\frac{4}{5}$). At the time, we believed this mistake stemmed from a mathematical reality that did not include fractions as quotients (especially because Justin was unable to explain his solution). However, we were now confronted with a more complicated situation.

These two students recognized that division was the appropriate operation on all five problems. However, each of them divided both ways on four of the five problems. For the problems in which they divided both ways, they always chose the correct answer, even though there is no evidence that the correct direction was their first instinct (Robert divided in the correct direction first on 25% of his "both ways" solutions and Justin divided in the correct direction first of 50% of his "both ways" problems). This suggests that Justin and Robert recognized division situations from the structure of the problem, but that they were choosing the *direction* of the division *ex post* based on the quotient, rather than on the structure of the problem.

Justin and Robert's "divide both ways" method accounted for some of the "multiple operations" that we observed. However, we also observed that many students performed other operations, even though they usually settled on the correct division as their final answer. For example, Protocol B shows how two other students used multiple operations when solving a problem involving sharing 546 candies amongst 13 people.

i.	ii.
Calculator: 546-13=533	Calculator: 13*546=7098
Calculator: CLEAR	Calculator: CLEAR
Calculator: 546/13=42	Calculator: 546/13=42
Types: 42	Types: 42

Protocol B Two examples of how students used multiple operations to solve a problem that involved sharing 546 candies amongst 13 people

This suggests that although students almost always settled on the correct division, it was not necessarily their first instinct. For some students, the problem did not indicate division right away. For others, the problem indicated division, but not the direction of the division.

Summary: Analysis of research question 1

Our first research question was: "How do our students solve partitive division problems with integral and non-integral results?" Our analysis suggests that the mathematical realities of our students did not include strategies for recognizing division situations or for determining the direction of the division within a division situation. Furthermore, it suggests that pre-formal productions are key components of the mathematical realities of students who solve fair sharing problems correctly, but that such productions are not real for most students.

This analysis formed our "instructional starting point" for the learning path that emerged as we explored our second research question.

Exploring research question 2

The first two learning activities gave us a good sense of our students' mathematical realities around partitive division problems and the fraction-as-quotient sub-construct. Our second research question involved an intervention to see how students reinvented new productions through mathematical activity. To explore this question, we engaged students in a series of activities, designing each learning activity only after analyzing the results of the previous activity. Figure 5 shows a schematic outline of the learning path that emerged from this process. As shown, Learning Activities 3-5 initially involved fair sharing situations, while Learning Activities 6 & 7 involved division, and linked division to fair sharing. These activities are described below.

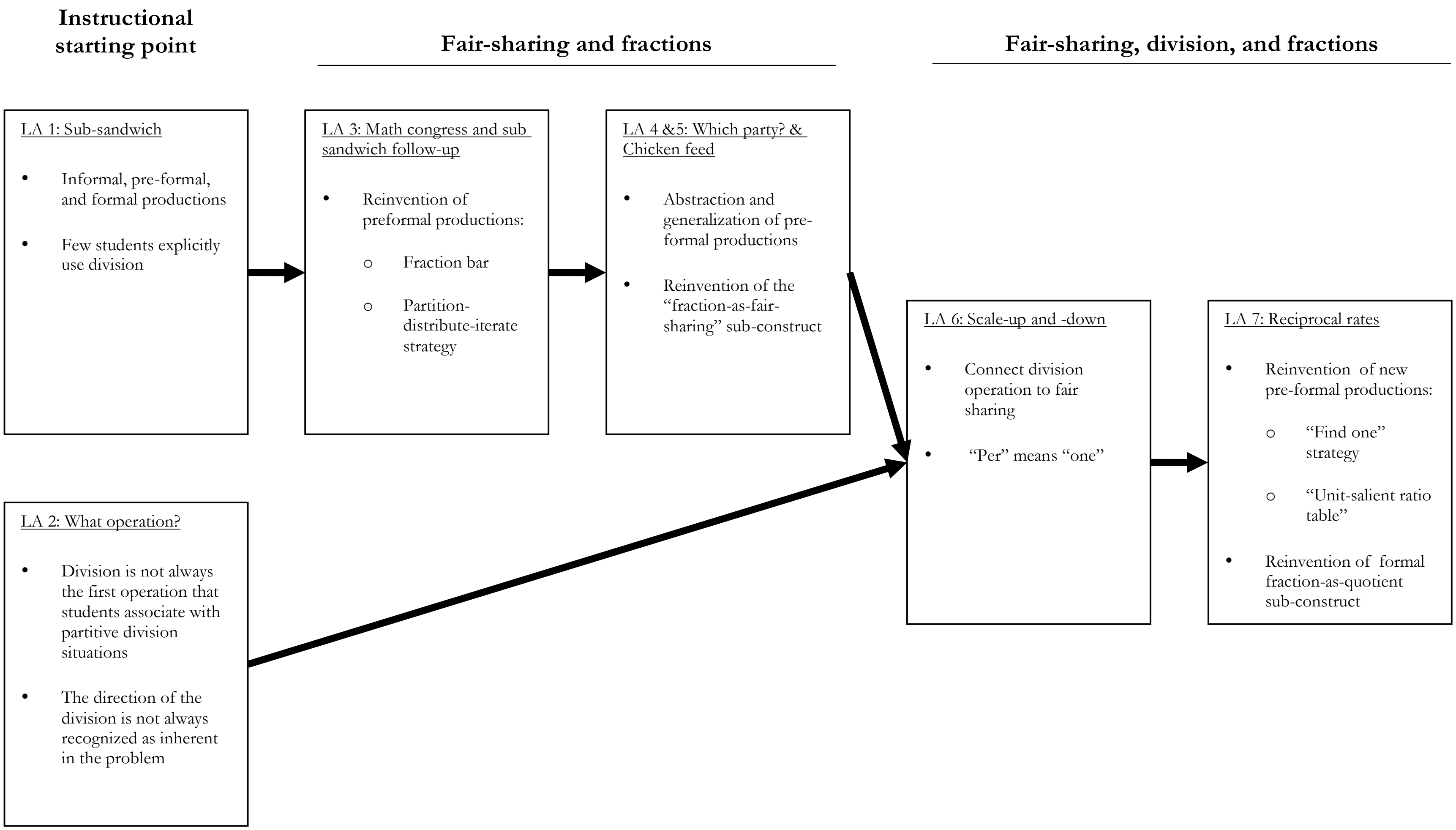


Fig. 5 A schematic of the path of learning. Each box represents a Learning Activity (LA), and the bulleted points within boxes represent the key outcomes of the learning activity. Learning Activities are arranged in temporal sequence from left to right. Bolded arrows represent ideational inheritance (i.e., LA 3 was informed by LA 1, whereas LA 6 was informed by LA 2 and LA 5)

Learning activity 3: Math congress and sub-sandwich follow-up problem.

Motivation and design

Our analysis of Learning Activity 1 suggested that pre-formal productions are key components of the mathematical realities of students who solve fair sharing problems correctly. We therefore set out to design a learning activity to help students reinvent these preformal productions.

In Learning Activity 1, two students had used preformal models and strategies to solve the sub-sandwich problem, and we wanted to create a learning activity in which these students could share their productions in such a way that they were rooted in the informal reality of the sub-sandwich problem. We therefore designed a "math congress" (Fosnot & Dolk, 2001, 2002), in which a carefully sequenced subset of students presented their solution strategies to the class.

Specifically, we planned to ask Penny, then Julia, then Theo to present. We wanted to begin with Penny because her informal *model of* the situation (Figure 2a) was rooted in the reality of actually cutting and distributing sub sandwiches to a group of people. We chose Julia to present next because her preformal model (Figure 2b) signified this concept, with squares representing sandwiches and shapes representing people. However, Julia's model still depicted the process of distributing pieces to people, thus rooting the abstraction in the informal reality of Penny' model. Theo also used a preformal model with different designs corresponding to different people, but his model (Figure 3) was even more abstract because he did not show the process of distribution. Finally, while Penny and Julia both used benchmark fractions in their informal precoordinating strategies, Theo used a preformal coordinating strategy. Thus, we felt that this sequence of students would help the class construct a chain of signification beginning in the problem context and culminating in Theo's preformal productions. Notice that Theo's strategy includes the first two pieces of the "partition-distribute-iterate" strategy, but Theo did not iterate the shares. A second goal of the math congress was therefore to reinvent this last piece of the strategy. During the math congress, we encouraged the use of iteration by asking the students questions about "how much each person received."

After the math congress, we had students work on a follow-up problem (Figure 6), based on "Group 3" from the initial sub-sandwich problem. We specifically focused the question on the *quantity* each sharer received in order to encourage iteration (Wilson et al., 2012).

Eight people shared seven sub sandwiches equally. How much of a sandwich did each person get?

Fig. 6 The "follow-up sub sandwich problem"

Analysis

During the math congress, Penny, Julia, and Theo each presented their work. Penny and Julia explained the partition and distribute strategy, and Theo explained his strategy for coordinating the partitions with the number of sharers. Following these presentations, Joshua explained how he could use iteration to quantify the share that each person received:

(1) FAP:	Joshua, how much-
(2) Joshua:	Four fifths
(3) FAP:	Where do you see four fifths?
(4) Joshua:	From each sandwich he is going to give him one fifth. One-fifth plus one-fifth plus one-fifth plus one-fifth

By sharing their productions, Penny, Julia, and Theo helped to create a chain of signification that included two preformal productions: bar models for fractions and a coordinating strategy for partitioning. These preformal productions then mediated the reinvention of new productions. Recall that in Learning Activity 1, Joshua wrote a formal fraction ($\frac{4}{5}$) to quantify the amount that each sharer received, which he supported with a single bar model with four-fifths shaded. As shown in the segment of talk above, once the bar model and coordinating strategy emerged in class, rooted to the informal reality of the sub-sandwich problem, Joshua's thinking about $\frac{4}{5}$ changed from a quantity that represented 4 out of 5 pieces in a single bar to a quantity composed of four one-fifth pieces, scattered across multiple bars. Thus the preformal productions mediated Jonathan's reinvention of iteration, the last piece of the partition-distribute-iterate strategy.

Learning activities 4 and 5

Motivation and design

These learning activities were motivated by stage 2 in our hypothetical learning trajectory. In this stage, we wanted students to mathematize progressively more formal and abstract fair-sharing situations. We hypothesized that such mathematization, along with social interaction organized around students sharing their productions with each other, would lead students to construct progressively more formal mathematical realities. As shown in Figure 7, the items to be shared in Learning Activities 3 and 4 became progressively more abstract as compared to the sub sandwiches that students initially shared. In Learning Activity 3, the items were bottles of liquid, which, although still shaped like a bar, cannot physically be cut like sub sandwiches. Hence applying the bar model to a bottle of liquid is more abstract than applying this model to a sub sandwich. In Learning Activity 4, the objects to be shared were "pounds of chicken food." These are shapeless, and hence using a bar model to model a pound is still more abstract.

Learning activity 3: Which party?	Learning activity 4: Chicken feed
You have a choice between two parties. One party will have six bottles of coke for eight people. The other party will have four bottles for five people. If you want the most Coca-Cola in your cup, which party would you choose to go to?	Mr. Huang owns 7 chickens. He gives the same amount of food to each of them. Yesterday, he fed his chicken 4 pounds of food all together. How many pounds of food did each chicken get?

Fig. 7 Learning activities 3 and 4

Analysis

As in Learning Activity 1, students used productions from all levels as they worked on Learning Activities 3 and 4. However, by Learning Activity 4, all of the students used preformal productions. As we discuss below, these productions mediated the students' activity and they also meditated the reinvention of more formal productions.

For example, in Learning Activity 4 a student initially used an informal precoordinating strategy, distributing ½ of a pound to each chicken. She soon realized that she would have ½ lb. left over. Importantly, however, when she experienced the perturbation (von Glasersfeld, 1989) caused by the remainder of ½, she could access a more general strategy, namely a preformal coordinating strategy. This is an example of how preformal productions mediated activity.

In other cases, preformal productions mediated the reinvention of more formal mathematical productions, by helping to make the connection between fair-sharing and formal fractions real for students. For example, in Learning Activity 4, Joshua solved the problem correctly using formal notation ($\frac{4}{7}$) without any supporting work. This was very similar to his solution in Learning Activity 1, which was also a correct bald fraction. However, this time, when FAP asked him to explain his work, Joshua used the preformal bar model and the preformal partition-distribute-iterate strategy to show why each chicken got $\frac{4}{7}$ of a pound.

Thus, through mathematical activity and social interaction, students created a chain of signification, culminating in robust preformal productions for fair-sharing. These productions mediated activity and they also mediated the reinvention of more formal productions. Indeed, in Learning Activity 4, nearly half of the students initially used formal fraction notation to express their answers.

Even as these formal notions of fractions were being invented, at one point or another in Learning Activity 4, many of the students either wrote a reciprocated fraction (i.e. $\frac{7}{4}$), or modeled the problem "backwards" (i.e., they drew seven fraction bars and distributed them to four sharers). In all cases, students were able to catch their mistake by moving backwards through the chain of signification. For example, a group of students initially wrote the reciprocal of the correct fraction (i.e., $\frac{7}{4}$). When FAP asked them to explain their work, they explained that the seven represented the number of chickens and the four represented the number of pounds. From FAP's perspective, they had essentially just explained that they had found the number of chickens per pound, rather than the number of pounds per chicken. Without indicating that their fraction was incorrect, FAP asked the students to draw a picture to justify their answer. Despite their reciprocated fractions, the students drew four bar models, which they explained represented the four pounds of food. They then proceeded to use the partition-distribute-iterate strategy to conclude that in fact, each chicken received $\frac{4}{7}$ of a pound of food. Thus, the students were able to catch their mistake by reasoning from a formal fraction to its real-world referent *via* preformal productions.

Why were the preformal productions necessary? Recall that when the students initially justified their answer they referred to the referent of the numerator separately from that of the denominator, without linking the two units (for example, by using the word "per"). It therefore seemed that these students perceived the units of the numerator and denominator as two *extensive quantities*, and not as two components of a single *intensive quantity* (e.g., "pounds per chicken;" see Nunes et al., 2003; Schwartz, 1988). We realized that while the preformal bar model and partition-distribute-iterate strategy helped students reinvent formal fraction notation as the outcome of fair sharing (henceforth referred to as the "fraction-as-fair-sharing" sub-construct), these productions did not lead to a reality in which the fraction was perceived as an intensive quantity or as a result of a division operation.

We hypothesized that understanding the notion of an intensive quantity might help students (a) associate fair-sharing with division, and (b) perform the division in the correct direction. This is because intensive quantities are *created* by division. The nature of these quantities depends on two factors: (1) the units of the dividend and of the divisor, and (2) the direction of the division. For example, in the chicken feed problem, the two units are chickens and pounds. Through division, it is possible to create two different intensive quantities: "pounds per chicken," and "chickens per pound." The operation of division creates the intensive quantity, and the direction of the division determines the reference quantity (i.e., "chickens" in the intensive quantity "pounds per chicken"). Thus, we felt that if students could recognize a situation in which the final answer will be an intensive quantity, they could use this to recognize that division is an appropriate operation as well as the direction of the division. This led to the design of our next two learning activities.

Learning activity 6: Scale up and down

Motivation and design

Our goal for this learning activity was to help students reinvent the notion of an intensive quantity, and to associate this quantity with the division operation. One way that students can reinvent these ideas is through engaging in missing value problems that involve proportional reasoning with rate pairs (e.g., if 3 apples cost $0.90, how much does 1 apple cost?) (Clark, 2005; Cramer, Bezuk, & Behr, 1989; Post, Behr, & Lesh, 1988). Students use a variety of strategies when solving such problems (Clark, 2005). Incorrect strategies are often *additive*, and involve adding or subtracting the same numerical value to both extensive quantities in the rate (e.g., adding "1" to both the apples and the cost in the example above). The most basic correct strategy is the *build-up* strategy, which also relies on repeated addition, but with equal quantities being added to both extensive quantities, rather than equal numerical values (e.g., adding "1" to the apples and $0.30 to the cost; Lesh, Post, & Behr, 1988).

More advanced strategies involve multiplicative reasoning that maintain constant ratios *between units* and *within units*. For example, in the apple problem given above, a within-unit strategy would involve students recognizing that the number of apples has decreased by a factor of three, and hence the price should decrease by a factor of three. A between-unit strategy would involve students recognizing that in the first rate pair, the numerical price is 0.30 times as large as the numerical apples, and hence in the second rate pair, the numerical price should also be 0.30 times as large as the numerical apples. This terminology is muddled, with some authors using the terms "within" and "between" to refer to operations within and between *ratios*, instead of within and between *units*, effectively switching the meanings of within and between. To maintain clarity, we will use the terms "within-units" and "between-units." Under most circumstances, students prefer to use within-unit strategies (Karplus, Pulos, & Stage, 1983; Vergnaud 1988; G. Brousseau et al., 2008).

Using the language of progressive formalization, we consider strategies that rely on additive reasoning (the incorrect additive strategy and the build-up strategy) to be informal strategies because they are not successful across a wide variety of problems (a build-up strategy is not successful when the ratio between the rate pairs is non-integral; Kaput & West, 1994). We consider strategies that rely on multiplicative reasoning to be preformal strategies because they are general strategies that can be used successfully across problems. We felt that preformal multiplicative strategies would mediate the reinvention of division as a fair-sharing operation, so we wanted our learning activity to encourage these strategies. With this consideration in mind, we designed a problem string (Kindt, 2010) of missing value proportional reasoning problems (see Figure 8).

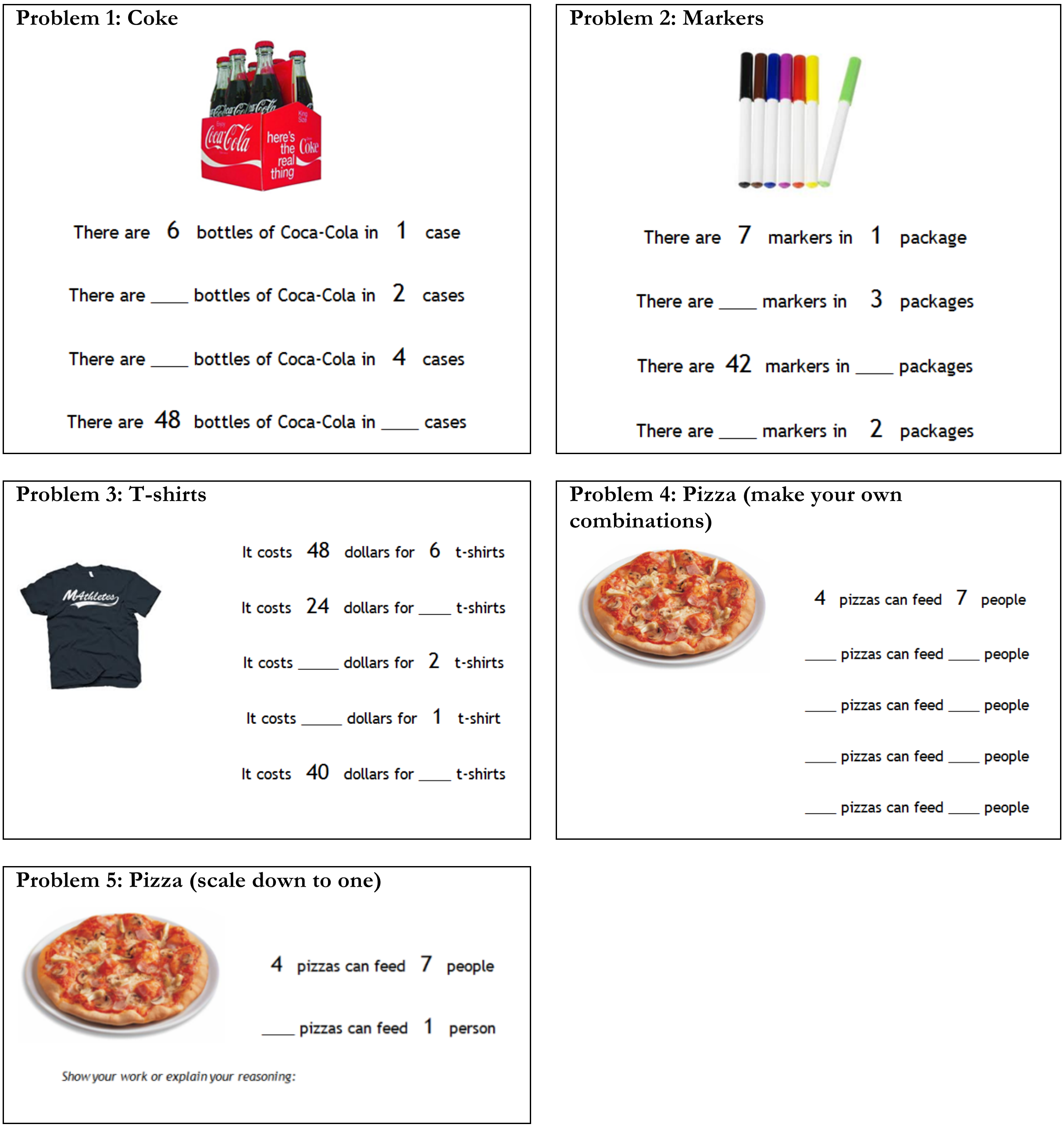


Fig. 8 The problem string for Learning Activity 6

As shown in Figure 8, the problem string is composed of the following sequence, which encourages the use of within-unit multiplicative strategies:

1. The *Coke problem* involves missing value problems that can be solved using a within-unit strategy by doubling the quantities in the rate pair directly above.
2. The *marker problem* requires students to multiply by other factors, or to consider rate pairs that are not immediately above the particular missing value.

3. The *t-shirt problem* requires division, first by two and then by other factors.
4. The *pizza: make your own combinations* problem allows students to make their own combinations in a situation where the within ratio is not an integer.
5. The *pizza: scale down to one* problem combines the proportional reasoning structure of the first four problems with a fair-sharing situation and a non-integral solution to the missing value.

We chose very familiar contexts for these problems in order to make the additive strategy unlikely (Karplus et al., 1983). Furthermore, we believed that the progression of the arithmetic required would encourage the progression from the informal build-up strategy to preformal multiplicative strategies, in particular the within-unit strategy.

This was important because the last problem in the string (pizza: scale down to one) combines the proportional reasoning structure of the first four problems with a fair-sharing situation and a non-integral solution to the missing value. We believed that if students used multiplicative reasoning (that is, division) to solve this problem, it would help them construct the association between division and fair-sharing, and the hence the association between the "fraction-as-fair-sharing" sub-construct and the fraction-as-quotient sub-construct.

Analysis

In the first problem (Coke), students used build-up and within-unit strategies. In the subsequent math congress, students presented both strategies, and the class seemed to coalesce around the within-unit strategy. For the second problem (markers), all students used the within-unit strategy. During the math congress, a serendipitous moment occurred: Jody used the word "per." FAP tried to capitalize on this moment, as shown below:

(1) Jody:	I looked at the seven and the three and thought "21". It is easier for me, a time saver… it's just easier… if there is three packages and seven per package
(2) FAP:	Oooh – that's a good word, per. What does "per" mean? (writes "7 markers per package" on the board)
(3)Lori (overlapping):	For every one package
(4) Joshua (overlapping):	For every one

This was an important moment because, as discussed above, we hypothesized that an understanding of intensive units could mediate students' activity in partitive division situations. This hypothesis relies on (a) students using the word "per" when stating the intensive units, and (b) understanding that "per" indicates "for every one."

During the math congress for the third problem (t-shirts) a *unit rate strategy* (Cramer & Post, 1993) emerged, and two students explained how they used division to find the unit rate. At this point, FAP prompted the class use "Jody's word" (per) to explain the unit rate. In the fourth problem (pizza: make your own combinations), only two students used a build-up strategy to make their own combinations, while the rest of the students used within-unit multiplicative strategies. Thus, the problem-string up to problem four, as well as social interaction encouraged reinvention of a within-unit strategy which then mediated subsequent activity.

In the final problem of the string (pizza: scale down to one), we were specifically interested in (a) whether students would associate the fair sharing situation with the division operation, and (b) the productions that students would use for fair sharing. All but one student indicated a division operation to solve this problem, although there was great diversity in the ways that students used division:

- *Within-unit division by seven,* which resulted in the correct answer. Students who used this strategy wrote used formal fraction notation to write their final answer, and used the partition-distribute-iterate strategy to justify their solutions.
- *Repeated halving*, in which students initially tried to repeatedly divide both the people and pizzas by two (i.e., a within-unit strategy of repeated halving) in order to get down to one pizza. When this did not work, students used a within-unit strategy of dividing both extensive quantities by seven.
- *Different divisors*, in which students initially divided the pizzas by seven and the people by four, which resulted in the statement, "1 pizza can feed 1 person." Students who used this strategy recognized the incongruence between this statement and the initial statement

(4 pizzas can feed 7 people), and either used a within-unit strategy of dividing both extensive quantities by 7, or the partition-iterate-divide strategy.

- *Divide both directions.* As in Learning Activity 2, Robert divided in both directions and used the results to choose his answer. As shown in Figure 9, Robert indicated a correct division using a between strategy, but then used long division to divide seven by four *and* four by seven. Upon examining the decimal results, he chose the incorrect answer because "it looks easier" and rejected the correct answer of 0.57142 because one "can't split a pizza into that."

Thus, by the end of the sequence, nearly every student saw the division operation in a fair sharing problem. Furthermore, many students linked the division operation to the partition-distribute-iterate strategy. Finally, students continued to formalize the fraction-as-quotient sub-construct: many students wrote the correct fraction without using the preformal partition-distribute-iterate strategy to find it (although some of these students used this strategy to justify their solutions, again showing the chain of signification in which the formal notion was built on the preformal strategy).

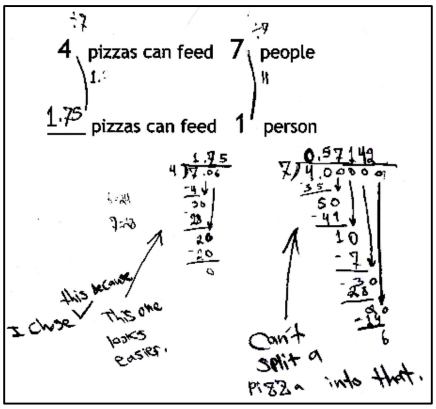


Fig. 9 Robert divided both ways, and chose the incorrect answer because it looked easier

However, we also observed a number of mistakes in the way students used division on the final problem. Our next task was to design a learning activity that would continue to help students (a) associate fair sharing with the division operation, and (b) recognize how the units of the final answer can be used to determine the dividend and divisor.

Learning activity 7: Reciprocal unit rates

Motivation and design

At this point, students had created a chain of signification for fair-sharing, which included preformal productions and culminated in a "fraction-as-fair-sharing" construct. Furthermore, many students were linking fair-sharing to division, and hence formalizing the fraction-as-quotient sub-construct. However, this trajectory was obstructed for some students because either (a) they were making mistakes when using division to solve fair-sharing problems (using incorrect divisors or reciprocating the dividend and the divisor), or (b) they had not-yet constructed the link between fair-sharing and division.

We felt that if students could recognize situations where an intensive quantity was the final answer, they could use this to recognize the necessity of dividing two extensive quantities, as well as the divided and divisor of this division operation. However, this relies on students reinventing the division operation as a referent-changing operation (Schwartz, 1988), namely one that transforms two extensive quantities into an intensive quantity. Our goal for this learning activity was to help facilitate this reinvention.

To do this, we designed a two-problem string to explicitly connect the division operation to the creation of an intensive quantity. As shown in Figure 10, the problems invoke a fair-sharing context and have the same structure as the problems in Learning Activity 6. This was to help students associate division with fair-sharing. Furthermore, both problems share the same antecedent, and both problems ask students to create an intensive quantity that can be interpreted as a unit rate. However, the two unit rates are reciprocals of each other. This was to help students associate the direction of the division with the compound units of the intensive quantity that is created. Our choice of dollars and pizza for the antecedents was purposeful for two reasons. First, we wanted to use quantities that were both "partitionable," but which still invoked a fair-sharing context. Second, we wanted the intensive quantity (e.g., "pizza per dollar") to feel new to students, as if they *created* it using division. We did not want to use familiar quantities such as speed or density because students do not always associate division operations with insensitive quantities that they are familiar with (e.g., Thompson, 1994).

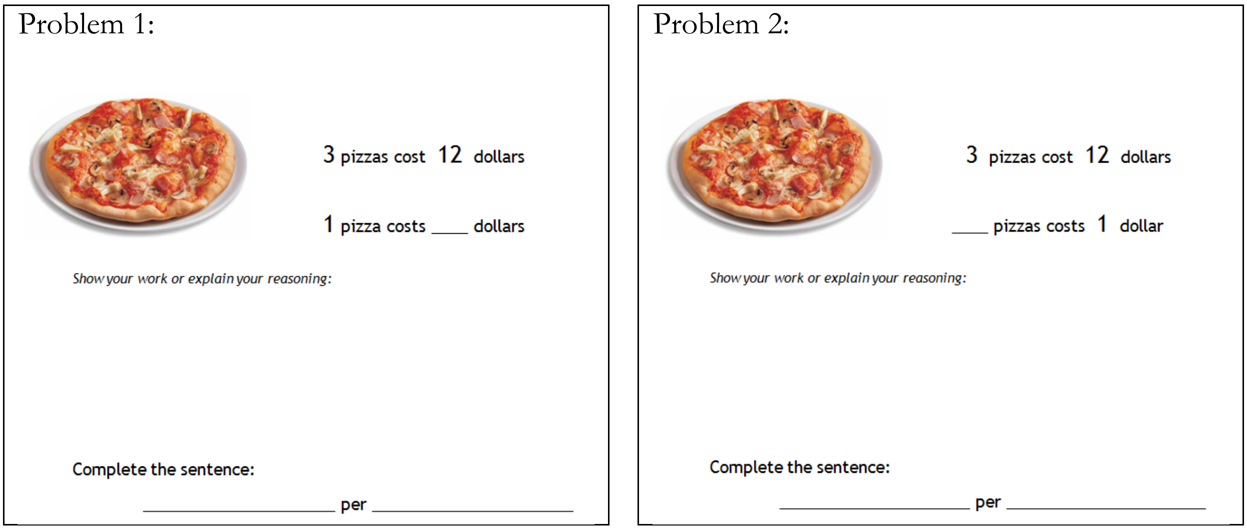


Fig. 10 The problem string for "reciprocal unit rates"

Analysis

Our analysis of this problem centered on three aspects: (1) How students used division, (2) how students used units, and (3) the continued formalization of the fraction-as-quotient sub-construct.

How students used division: All students used a within-units strategy and indicated division by three for problem one and division by 12 for problem two. None of the students divided both ways before choosing a final answer. This represented a significant change from Learning Activity 6. For example, Lori initially used repeated halving in Learning Activity 6, but she chose a more strategic divisor in Learning Activity 7 (Figure 11).

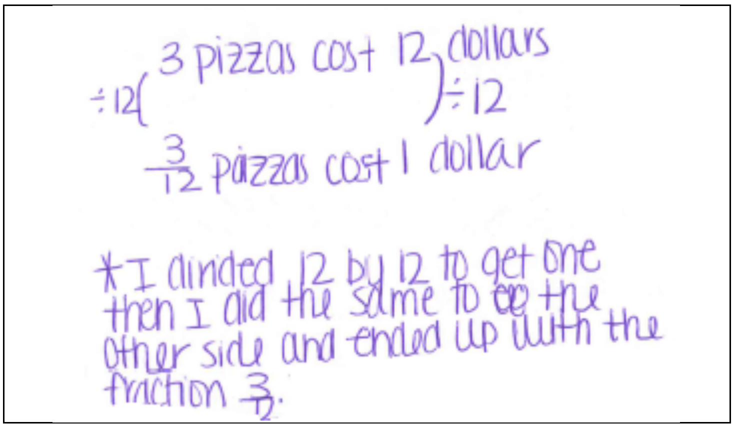


Fig. 11 Lori chose a strategic divisor in Problem 2

Furthermore, recall that Robert divided both ways in Learning Activity 6, and relied on the decimal representation of the division to guide him to an (incorrect) answer. In contrast, in Learning Activity 7, Robert did not divide both ways. Rather, he recognized the correct direction of the division on both problems, and he did so *before* dividing (Figure 12). He explained to FAP that he knew which direction to divide based on which quantity he was finding one of.

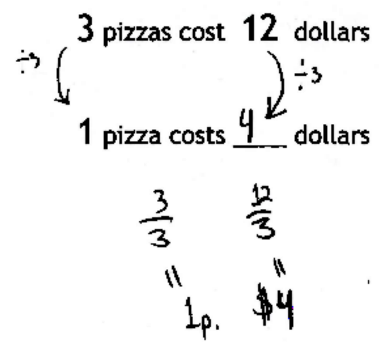
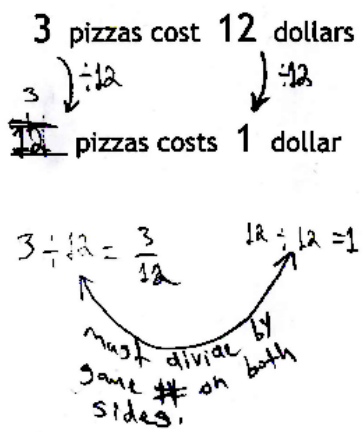
Problem 1:	Problem 2:
	

Fig. 12 Robert divided in the correct direction on both problems. He explained to FAP that he knew which direction to divide based on what he was trying to find one of

We had hypothesized that that the strategic use of units might help students understand how the direction of the division is inherent in the problem structure, namely "If students can state the units of the final answer as a compound unit using the word 'per' before they solve the problem, they will be able to recognize partitive division situations and the direction of the division." However, based on the student work, it seemed that students were not employing this "referent-transforming strategy." Students were considering the units in the problem in a thoughtful way, but not in the way that we hypothesized.

How students used units, and the emergence of new preformal productions: Consider problem one, which involves finding the missing dollar value that corresponds to one pizza. We hypothesized that students would recognize this as a situation in which they wanted to find the intensive quantity, "dollars per pizza," and that, by stating the units in this way students would recognize the division of dollars by pizza.

As the learning activity progressed, however, it became clear that this is not how students thought of the missing value problem. Rather than using compound units to construct a division operation, students were choosing the dividend and divisor based on which of the extensive quantities they were trying to find one of. This "find-one" strategy turned out to be the dominant strategy. Rather than dividing dollars by pizzas to arrive at the intensive quantity "dollars per pizza," students were dividing both dollars and pizzas by a dimensionless scalar that maintained the units in each extensive quantity, and then writing the "per" statement based on which quantity they had "found one" of. It was not clear that students recognized their result as a new quantity, nor was it clear that students had constructed the division operation as a referent-changing operation.

We believe that the visual design of the missing value problems supported students in their use of the "find-one" strategy. The vertical alignment of units helped students recognize the within-unit scalar needed to "find one," and the white space between the lines encouraged students to draw arrows showing division as a dynamic operation that transformed a units into 1 unit. With respect to the arrows themselves, consistent with G. Brousseau et al. (2009), we see the students' use of arrows as a way of reasoning, "whose validity the student checks by reference to the actual meaning" (p. 109). We did not dwell on the arrows themselves in the classroom (G. Brousseau et al., 2009 suggest that this could lead to a "meta-didactical slippage"), but rather asked students to explain their mathematical reasoning in the context of the problem itself (e.g., the explanations in Figures 11 and 12).

Students seemed to find the vertically-aligned structure of the problems so helpful that they even reproduced it themselves (see, e.g., the student work in Figure 11). Thus this structure soon became a preformal tool that students used to solve missing value proportional reasoning problems. As such, it is essentially a ratio table (Streefland, 1993; Middleton & van den Heuvel-Panhuizen, 1995). One key difference is that the units are made more salient in this tool, and hence, the division operation needed to "find one" is clearer. Because this tool keeps the units salient, we called it a "unit-salient ratio table."

The continued formalization of the fraction-as-quotient sub-construct: When solving problem 2, students were confronted with a division problem with a non-integral solution. In this problem, all students recognized that $3 \div 12 = \frac{3}{12}$, indicating a formalization of the fraction-as-quotient sub construct (see Figures 11 and 12 for examples of how two students demonstrated this formalization).

Summary: Analysis of research question 2, and the importance of preformal productions

Our second research question was, "how do beginning algebra students reinvent the fraction-as-quotient sub-construct?" Our hypothetical learning trajectory involved relatively unproblematic movement through a progressive-formalization sequence: students would initially use informal productions to solve fair sharing problems, and—through mathematizing progressively more abstract situations and social interaction—students would invent progressively more formal mathematical productions culminating in the formal fraction-as-quotient sub-construct. What we found was that the instructional starting point was considerably more complicated, and that preformal productions played a larger role than we had initially expected.

First, we found that our students initially solved fair-sharing problems using productions at all levels formalization, but that only those students who used preformal productions could explain their reasoning and solve the problem correctly. This was our initial clue of the importance of preformal productions. As students shared their productions and mathematized more abstract situations, we found that preformal productions played two key meditational roles: (1) they mediated problem-solving activity, allowing, for example, students to solve the chicken feed problem in Learning Activity 4 after they realized that their informal strategy would not lead to equal shares; and (2) they mediated the reinvention of more formal productions, resulting in a chain of signification that made the fraction-as-quotient sub-construct *real* for students. Thus, preformal productions were important even for students who initially solved the sub-sandwich problem correctly using formal mathematics.

Second, we found that many students did not automatically associate partitive division situations with the division operation, and, even when students did use division, they sometimes performed the division operation both ways and compared the results to choose their final answer. Many times, this strategy led students to choose the correct answer (e.g., Robert and Justin in Learning Activity 2), but other times it did not (e.g. Robert in Learning Activity 6). Again, preformal productions—specifically the unit-salient ratio table and the find-one strategy—played a key role, mediating the reinvention of division as a “find one” operation.

In contrast to the (expected) preformal productions that emerged for fair sharing, the unit-salient ratio table and the find-one strategy for division emerged in class unexpectedly. We now recognize that the “referent-changing” strategy that we had hoped to design for is a between-unit strategy, and we designed for a within-unit strategy. In one sense, this is a failure of design, and we highlight it so as to provide an avenue for future design work. Despite this, students reinvented a robust strategy that they could use to recognize when and how to divide (G Brousseau et al., 2004, describe a similar strategy used by students in their study to find the thickness of a single sheet of paper). Thus, although students did not construct the strategy that we expected, we still consider outcome of the learning activity a success. In fact, we believe that these productions could play a powerful role in an algebra class. We did not have time to explore this in depth in our design experiment, but we have some evidence from the class. During a review day for the final exam at the end of the school year, we gave students the problem shown in Figure 13. In the figure, the student coordinates the "slope triangle" with a unit-salient ratio table and the find-one strategy. This leads us to conjecture that the unit-salient ratio table and find-one strategy could mediate students' reinvention of slope as a unit rate. To be clear, the student whose work is shown in Figure 13 had already learned slope by the time we gave this problem. Thus, we present this work only as a suggestion of potential. We plan to explore this conjecture in our future work.

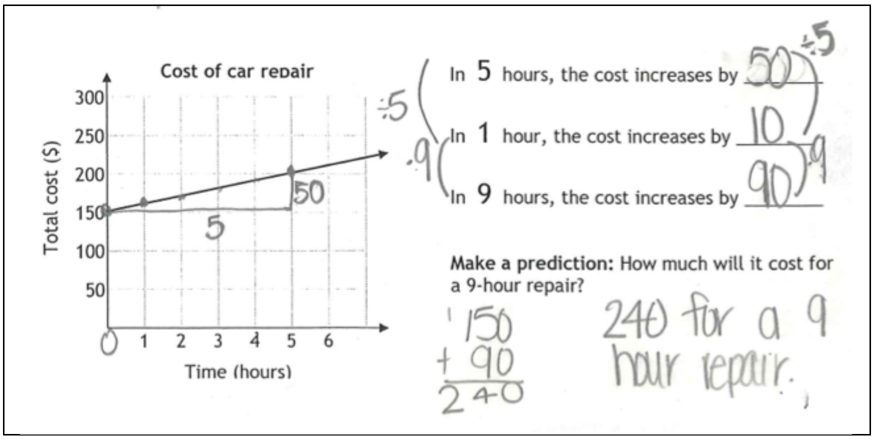


Fig. 13 How the find-one strategy and the unit-salient ratio table might mediate students' reinvention of slope as a unit rate

A closer look at preformal productions

In the discussion above, we explained how preformal productions played two key roles for our students: (1) they mediated mathematical activity, and (2) they mediated the reinvention of more formal mathematical productions. We further showed how sometimes, preformal productions emerge in unexpected ways. In this section, we take a closer look at preformal productions, examining how they emerged in our classroom and reflecting on their epistemological and ontological status. We suggest that preformal productions can be designed for, and that that even when preformal productions emerge unexpectedly, they are not random. In both cases, preformal productions embody historic classroom activity and social interaction. As such, we define preformal productions ontologically as cultural artifacts.

Preformal productions can be designed for

Our goal for Learning Activities 3-5 was to design a sequence of activities and social interactions that would lead students to construct the preformal bar model and partition-distribute-iterate strategy. Following RME design principles (Gravemeijer, 1999), we explicitly designed math congresses and mathematical activity to support students' reinvention of progressively more formal mathematical productions. For example, we structured the math congress in Learning Activity 3 such that the student presentations followed a trajectory of progressive formalization. This notion of a progressive formalization trajectory also informed the design of Learning Activities 4 and 5. As students engaged in these activities, the preformal bar model and partition-distribute-iterate strategy came to embody the historic activity of fair sharing and the social interaction of the math congress. That students reinvented these productions "as hypothesized" suggests that preformal productions can be designed for by designing activity and social interaction around a progressive formalization trajectory.

Preformal productions emerge unexpectedly, but they are not random

In contrast to the preformal productions that emerged for fair sharing, the unit-salient ratio table and the find-one strategy for division that emerged in Learning Activities 6 & 7 were unexpected. However, this does not mean that they emerged randomly. Instead, we suggest that, like the designed-for productions discussed above, these preformal productions also embody historic classroom activity and social interaction. Below, we justify this claim for the find-one strategy.

Preformal productions embody historic classroom activity

Partitive division contexts can be sub-divided into the (non-mutually exclusive) contexts of fair-sharing on the one hand, and finding unit rates on the other. In a fair sharing problem, the quotient represents the extensive quantity that one sharer receives, and in a finding unit rate problem the quotient represents an intensive quantity (Confrey et al. 2009 call the former a "many-as-one" conception and the latter a "many-to-one" conception). Of these two sub-contexts, the fair-sharing sub-context is more focused on the concept of "finding one" (i.e., how much pizza one person receives) as opposed to a unit rate problem (i.e., find the speed in miles per hour). To be sure, one can conceptualize the result of a fair-sharing problem as a unit rate (i.e., conceptualizing the aforementioned quantity of pizza as "pizza per person"): this, in fact, is how students wrote their final solutions in Learning Activity 7. The key here is the framing of the task: Are we looking for the (extensive) amount of pizza that corresponds to one person, or are we looking for a brand new (intensive) quantity, "pizza per person?" The former framing suggests a find-one strategy, while the latter farming suggests a "referent transforming" strategy. In the case of our experiment, all of our tasks were framed in the "find one" context. Thus, the students' historic classroom activity was strongly suggestive of the find-one strategy.

Preformal productions embody historic classroom social interaction

Social interaction also played a role in shaping the find-one strategy. To explore how social interaction shaped the find-one strategy, we analyzed the classroom discourse that MM recorded in his fieldnotes. In analyzing discourse, we were guided by the notion that discourse is not just communication about action, it is itself action. That is, discourse communicates, but it also *does* (Gee, 2011; Jaworski & Coupand, 2006). These two uses of discourse are mutually-implicated in interactive sequences, in which "we produce action methodically to be recognized for what it is, and we recognize action because it is produced methodically in this way" (Heritage & Clayman, 2010, p. 10). Actions are produced and evaluated in *turns* at talk. Thus, we analyze the work of discourse by analyzing how turns are *produced*, *taken-up*, and *sequenced* in interaction. In what follows, we analyze two sequences to explore how social interaction shaped the find-one strategy.

First, recall the classroom discourse that occurred when Jody used the word "per" in Learning Activity 6:

(1) Jody:	I looked at the seven and the three and thought "21". It is easier for me, a time saver… it's just easier… if there is three packages and seven per package
(2) FAP:	Oooh – that's a good word, per. What does "per" mean? (writes "7 markers per package" on the board)
(3) Lori (overlapping):	For every one package
(4) Joshua (overlapping):	For every one

In the first turn, Jody uses the word "per" to create an intensive quantity, "markers per package," without fully articulating the units. FAP writes the full units of the quantity on the board, but rather than attend to this new *quantity*, FAP attends to the new *word*, "per," and Lori and Joshua focus on the "oneness" of this word.

This focus on the oneness of "per" continued in the math congress held after students worked on the next problem in the string (t-shirts, see Figure 8). In the first part of this problem, students were given the cost of six shirts, and asked to find the cost of three shirts. We expected that students would recognize a within-unit ratio of two, and use this to find the cost of three shirts. However, Zane explained that he multiplied "eight times three" to find the cost of three shirts. We pick up the discourse as Toni attempts to explain Zane's strategy to the class:

(1) Toni:	He divided 48 by six, which is eight.
(2) FAP:	That eight is an interesting number. What does the eight represent?
(3) Toni:	How much one shirt costs
(4) FAP:	What was Jody's word?
(5) Many:	Per
(6) Steve:	It means one

Here, Toni starts by describing calculations on numbers, rather than calculations on extensive quantities. That is, she describes dividing the number 48 by the number 6, rather than on dividing the quantity "48 dollars" by the quantity "6 shirts." Thus, while Toni has explained how Zane found the number 8, it is not yet clear how she conceptualizes of this number. When FAP asks about the meaning of her number, she focuses on the oneness of it, and the subsequent discourse continues to reify the oneness of "per." Again, we can imagine a different situation where, instead of asking about the meaning of the "8" in turn 2, FAP asked about the meaning of the 48 and the 6, and then asked what happens when you divide dollars by t-shirts. Perhaps the notion of division as a referent transforming operation may have come out of the subsequent classroom interaction.

Thus, the find-one strategy emerged unexpectedly in our experiment, but it was not random. Instead, this strategy embodies historic classroom activity (including solving fair sharing problems) and social interaction (including discourse that focused on the oneness of the word "per").

Preformal productions are cultural artifacts

Having analyzed at some depth the role of preformal productions and their emergence in the classroom, we now turn to a more fundamental question: "what are preformal productions?" Drawing on the RME literature in our conceptual framework, as well as our expanded use of preformal productions in this paper, we define preformal productions as *mathematical productions—such as models, tools, and strategies—that embody historic activity and social interaction. They are simultaneously general and specific, and as such they exist between students' informal realities and formal mathematics. Through activity, preformal productions can be made general enough so as to be applicable to a wide variety of problems, but they retain contextual cues to specific situations.*

In the definition above we referred to the "existence" of preformal productions. In what sense do these productions exist? One response is to think of preformal productions as having an epistemological existence. From this perspective, we might consider G. Brousseau's distinction between two types of knowledge: *connaissance* and *savoir*. *Connaissances* are an individual's internal ways of knowing within a situation, while *saviors* are the culturally-accepted ways of expressing and communicating these ways of knowing:

> Events in class have the effect of provoking students to react, make declarations, reflect, and learn, all of these manifesting their intellectual activity. This activity reveals their *connaissances*: what they do, their intentions, their perceptions, their decisions, their beliefs, their language, their reasoning. Only one part of this set of *connaissances* is recognized as

> expressible, and expressed, whether by the student, by other students, by the teacher, or by society. These *connaissances* are recognized with the help of a repertory of *reference connaissances*: custom, language, rules of orthography, established definitions and theorems, logic, communal beliefs, culture, etc. These are the *savoirs*. *Savoirs* are the indispensable means of recognizing and expressing *connaissances* (G. Brousseau et al. 2009, p. 110)

Preformal productions seem to have elements of both *connaissances* and *saviors*. For example, the partition-distribute-iterate strategy may be considered a way of knowing, and thus a *connaissance*. On the other hand, the bar model may be considered a *savoir* because, within the classroom, it was the culturally-accepted external means through which students expressed their ways of knowing (in our case, the bar model become culturally-accepted within the classroom through the math congress described in Learning Activity 3). However, we feel that this does not quite capture the nature of preformal productions. First, notice that, tacit in the excerpt above is a definition of *culture* as a static entity that is established *a priori* and which exists to make the internal external. The implication is that the relationship between individuals and their culture is a one-way relationship: from internal to external, via culture. To capture the nature of preformal productions, we need a more dynamic definition of culture and a different conception of the relationship between individuals and their culture.

We define culture as both a process and a product. As a process, culture accumulates the prior accomplishment of a social group and propagates them into the present (Hutchins 1995). As a product, these accumulated accomplishments of history serve as resources for current activity (Cole, 2010). These resources are *cultural artifacts*, and they function as follows:

(1) *The historical nature of artifacts can be understood on many timescales:* "Some artifacts are inherited (e.g., cultural tools, methods, signs, software tools) and practically do not change during activities. Others are in perpetual transformation. They have been

freshly created as outcomes of previous actions and are used by the participants in further activities" (Schwarz & Hershkowitz, 2001, p. 251).

(2) *Cultural artifacts mediate current activity:* "Higher mental functions are by definition culturally mediated. They involve not a direct action on the world, but an indirect action, one that takes a bit of material matter used previously and incorporates it as an aspect of action. Insofar as that matter itself has been shaped by prior human practice (e.g., it is an artifact) current action incorporates the mental work that produced the particular form of that matter" (Cole and Wertsch, 1996, p. 252)

(3) *The relationship between persons and artifacts is bi-directional:* "[A]gents create meaning by drawing on cultural forms as they act in social and material contexts, and in so doing produce themselves as certain kinds of culturally located persons while at the same time reproducing and transforming the cultural formations in which they act" (O'Connor, 2003, pp. 61-62)

We suggest that the notion of a cultural artifact as defined above best-captures the nature of preformal productions. As we described above, preformal productions embody historic classroom activity and classroom interaction, and they were used by students in service of current activity. Thus, they meet our initial definition of a cultural artifact. Preformal productions fit the remaining functions of artifacts as follows: (1) Students reinvented preformal productions on the timescale of the design experiment. (2) Preformal productions played two mediating roles: They mediated students' mathematical activity, and they mediated students' reinvention of more formal mathematical productions. (3) In the combination of (1) and (2) above, we see the bi-directional nature of the relationship between students and preformal productions: Students acted to produce preformal productions, but preformal productions acted back to produce students as cultural beings with particular mathematical realities.

Conclusion

We began this paper with a problem from our practice as algebra teachers and researchers, namely, that beginning Algebra I students had more trouble with the "division step" in an algebra equation when the quotient was non-integral than they did when the quotient was an integer. In our practice, we observed that upon encountering such a division step, students often either (a) abandon the problem, stating that the division "can't be done" or (b) perform the division backwards. Furthermore, in our experience, nearly all students who attempt the division use the division symbol (÷) to represent the operation, and use the long division algorithm to express their final solution in decimal notation. Few students use the "fraction bar" (the line that separated *a* from *b* in the fraction $\frac{a}{b}$) to represent the division operation or use fractions to represent their solutions.

We hypothesized that this was because students did not have enough prior experience with the fraction-as-quotient sub-construct, and we conducted a design experiment organized around fair sharing in order to help students reinvent this sub-construct. In the beginning of the design experiment, we found that none of our students' initial realities included the fraction-as-quotient sub-construct. Furthermore, for many students the division operation was also problematic: students did not always associate partitive-division situations with the division operation, and, even when students did use division, they did not always see the how the problem situation dictated the direction of the division. By the end of the experiment, students had reinvented the fraction-as-quotient sub-construct, as well as the notion of using division to "find one" of a particular quantity. In describing how students reinvented these productions, we hope to have contributed to practice and to theory.

Our contributions to the practice of algebra teaching are (1) to highlight the importance of the fraction-as-quotient sub-construct for algebra students; (2) to suggest that this sub-construct, and indeed, the division operation itself, might not be a part of the mathematical realities for students entering Algebra I, and to have provided a detailed account of students' mathematical realities around division and fractions; (3) to show that students might not construct the fraction-as-quotient sub-construct solely through experience with fair-sharing situations, and that explicit activities may be needed to help these students link the "fraction-as-fair-sharing" sub-construct to the fraction-as-quotient sub-construct; and (4) to have provided one possible sequence of activities through which Algebra I students might reinvent fractions and division as they are used in algebra. In the beginning of the paper we clarified that our goal was not to present a "model" curriculum, and we reiterate this now. That said, we have shown that students learned powerful mathematics as they engaged in the sequence of activities presented here. As such, our descriptions of this sequence and the design decisions that motivated it may prove useful for teachers to design their own sequences. Future work should explore the ways in the preformal productions that students reinvented in this experiment mediate the reinvention of formal algebra.

Our contributions to theory include an expansion of the emergent modeling paradigm (Gravemeijer, 1999) to include all manner of mathematical productions, and an in-depth analysis of the role of preformal productions in (a) mediating students' mathematical activity and (b) mediating the reinvention of more formal mathematics. Luria (1928, p. 493) famously stated that "the tools used by man not only radically change his conditions of existence, they even react on him in that they effect a change in him and in his psychic condition." This is precisely the role played by preformal productions in our study. Preformal productions changed the conditions of our students' existence because they mediated students' activity, rendering solvable problems that were previously not solvable. Preformal productions further effected a change *in* our students because they mediated the invention of more formal mathematics. Indeed, the formal mathematical realities that emerged were—to a large extent—dictated by the preformal productions that preceded them. Preformal productions are not "crutches for the weak" as one teacher with whom we have worked once described them. Instead, they are an integral and vital part of doing and learning mathematics.

Given the importance of preformal productions, it is important to understand how they emerge in the classroom. We suggest that the specific productions that emerged in our classroom embody historic classroom activity and social interaction. As such, they can be designed for. However, we have also shown that preformal productions emerged even when they were not explicitly designed for. This analysis has large implications for designers and teachers. Namely, it suggests that preformal productions should be a key part of any designed curriculum, and that teachers should be aware of the ways in which the activity and interactions within the classroom are shaping the development of preformal productions, because it is on these productions that students create their formal mathematical reality.

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